Pg 1

Monday, October 14, 2019 2:50 PM

Section 11.6: Absolute Convergence and the Ratio and ~~Root~~ Tests

Definition: A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Definition: A series $\sum a_n$ is called **conditionally convergent** if the series $\sum |a_n|$ is divergent and the series $\sum a_n$ is convergent.

Theorem: If a series $\sum a_n$ is absolute convergent, then it is convergent.

Monday, October 14, 2019 3:07 PM

by the definition

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

Original

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

New series

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^3} \right| = \sum \frac{1}{n^3}$$

p-series $p = 3$

Conv.

Since the "new" series converges the original series is Abs. Conv.

by the definition

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

original series

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2+1}$$

by the def. of Abs conv. This sereies is Abs. conv.

new series ← conv.

$$\sum \left| \frac{\sin(n)}{n^2+1} \right| = \sum \frac{|\sin(n)|}{n^2+1}$$

$$|\sin(n)| \le 1$$

$$\frac{|\sin(n)|}{n^2+1} \le \frac{1}{n^2+1}$$

does $\sum \frac{1}{n^2+1}$ converge?

use LCT with $\sum \frac{1}{n^2}$ conv. $p=2$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

by LCT $\sum \frac{1}{n^2+1}$ will conv

by the comparison test $\sum \frac{|\sin(n)|}{n^2+1}$ will conv.

by the definition

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

new series

$$\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$$

p-series $p=1$ d.v.

new series does not conv. So this series is <u>not</u> Abs conv.

use AST (Alternating series test)

$b_n = \frac{1}{n}$

b_n dec

$\lim_{n\to\infty} b_n = 0$

$\sum \frac{(-1)^n}{n}$ will conv. by AST

Thus this series is conditionally conv.

Monday, October 14, 2019 3:07 PM

by the def.

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

new series

$$\sum \left| \frac{(-1)^n}{\ln(n)} \right| = \sum \frac{1}{\ln(n)}$$

use LCT with $\sum \frac{1}{n} \to$ div. $p=1$

$$\lim_{n\to\infty} \frac{\frac{1}{\ln(n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln(n)} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} n = \infty$$

by LCT $\sum \frac{1}{\ln(n)}$ div.

not Abs. conv.

Try AST

$b_n = \frac{1}{\ln(n)}$

$\lim_{n\to\infty} b_n = 0$ ✓

b_n dec ✓

by AST $\sum \frac{(-1)^n}{\ln(n)}$ will conv.

Thus $\sum \frac{(-1)^n}{\ln(n)}$ is cond. conv.

$\sum a_n$

The Ratio Test:

(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, with $0 \le L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: If the limit for the ratio test is 1, then this test fails to give any information. Try something else.

Consider the results of the ratio test for two of our known $p-$series.

$a_{n+1} = \frac{1}{(n+1)^2}$

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$, where $a_n = \frac{1}{n^2}$, converges since $p > 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)^2}{1/n^2}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1 \text{ by L'Hopitals.}$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n}$, where $a_n = \frac{1}{n}$, diverges since $p \le 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)}{1/n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1 \text{ by L'Hopitals.}$$

Ratio test will always fail for polynomials.

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use Ratio test for exp. factorial. n is in the power

Ratio test

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$$

$$a_n = \frac{(-3)^n}{n!} = \frac{(-1)^n 3^n}{n!}$$

$$\lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right|$$

$$a_{n+1} = \frac{(-1)^{n+1} 3^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} 3^{n+1}}{(n+1)!}}{\frac{(-1)^n 3^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 3^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n 3^n} \right|$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n\to\infty} \frac{3^n \cdot 3^1}{(n+1) \cdot n!} \cdot \frac{n!}{3^n}$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 5 \cdot 4!$$

$$= \lim_{n\to\infty} \frac{3}{n+1} = 0$$

Thus $\sum \frac{(-3)^n}{n!}$ is Abs. conv. by Ratio test

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} n!e^{-n}$$

$a_n = n!\,e^{-n}$ $\qquad$ $a_{n+1} = (n+1)!\,e^{-(n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,e^{-n-1}}{n!\,e^{-n}}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)\cdot \cancel{n!}\,\cancel{e^{-n}}\,e^{-1}}{\cancel{n!}\,\cancel{e^{-n}}} = \lim_{n\to\infty}\frac{n+1}{e^{1}} = \infty$$

by Ratio test This series will diverge

$2 \cdot n! = 2n! \neq (2n)!$

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(n+1)^3}{(2n)!}$$

$a_n = \frac{(n+1)^3}{(2n)!}$ $\quad a_{n+1} = \frac{(n+2)^3}{(2(n+1))!} = \frac{(n+2)^3}{(2n+2)!}$

$a_{n+1} \cdot \frac{1}{a_n}$

$$\lim_{n\to\infty} \left| \frac{(n+2)^3}{(2n+2)!} \cdot \frac{(2n)!}{(n+1)^3} \right|$$

$$\lim_{n\to\infty} \frac{(n+2)^3}{(n+1)^3} \cdot \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$\lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^3 \cdot \frac{1}{(2n+2)(2n+1)} = 1^3 \cdot 0 = 0$$

The series is Abs. Conv.

Pg 10

Monday, October 14, 2019 3:07 PM

$5(n+1)+2 = 5n+5+2$
$= 5n+7$

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(-4)^n n!}{7 * 12 * 17 * ... * (5n+2)}$$

$$a_n = \frac{(-1)^n 4^n n!}{7 \cdot 12 \cdot 17 \cdot \ldots \cdot (5n+2)}$$

$$a_{n+1} = \frac{(-1)^{n+1} 4^{n+1} (n+1)!}{7 \cdot 12 \cdot 17 \cdots (5n+2) \cdot (5n+7)}$$

Ratio test

a_{n+1} $\frac{1}{a_n}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} 4^{n+1} (n+1)!}{7(12)(17)\cdots(5n+2)(5n+7)} \cdot \frac{7(12)(17)\cdots(5n+2)}{(-1)^n 4^n n!} \right|$$

$$\lim_{n\to\infty} \frac{4^n \cdot 4(n+1)(n)!}{(5n+7)} \cdot \frac{1}{4^n n!} = \lim_{n\to\infty} \frac{4n+4}{5n+7} = \frac{4}{5}$$

The series will conv. by the Ratio test.

Example: The series $\sum a_n$ is defined recursively by

$$a_1 = 1 \qquad a_{n+1} = \frac{(2+\cos(n))a_n}{\sqrt{n}} \text{ for } n \geq 1.$$

Is the series is absolute convergent, conditionally convergent, or divergent?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{2+\cos(n)}{\sqrt{n}} \cdot a_n}{a_n}$$

$$= \lim_{n\to\infty} \frac{2+\cos(n)}{\sqrt{n}} = 0$$

By Ratio test the series is Abs. conv.

thus conv.