Pg 1

Monday, November 4, 2019 7:54 PM

Section 11.9: Representations of Functions as Power Series

Geometric Power Series: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$ converges for $|x| < 1$ with Radius of convergence $= 1$ and interval of convergence $(-1, 1)$

Building Block

$$\boxed{\frac{1}{1-x} = \sum_{n=0} x^n \qquad R = 1 \qquad I: (-1, 1)}$$

$|\square| < 1 \longrightarrow \frac{1}{1-\square} = \sum_{n=0} \square^n$

Example: Find the power series representation of $f(x)$ and the radius and interval of convergence.

$\sum_{n=0} x^n = \frac{1}{1-x}$ $R=1$ $|x|<1$ $I: (-1,1)$

A) $\frac{1}{4+x} = \frac{1}{4\left(1+\frac{x}{4}\right)}$

$= \frac{1}{4} \cdot \frac{1}{1 - \boxed{\frac{-x}{4}}} = \frac{1}{4} \sum_{n=0} \left(\frac{-x}{4}\right)^n = \boxed{\frac{1}{4}} \sum_{n=0} \frac{(-1)^n x^n}{4^n}$

$\frac{1}{4+x} = \sum_{n=0} \frac{(-1)^n x^n}{4^{n+1}}$

Can do the Ratio test to find Radius of conv. & then find the Interval of conv. like in Section 11.8

will converge if $\left|\frac{-x}{4}\right| < 1$

$\left|\frac{x}{4}\right| < 1 \longrightarrow |x-0| < 4 \longrightarrow R = 4$

$|x| < 4$

$-4 < x < 4 \longrightarrow I: (-4, 4)$

Monday, November 4, 2019 7:54 PM

B) $\frac{x^2}{4+x} = x^2 \cdot \boxed{\frac{1}{4+x}} = x^2 \sum_{n=0} \frac{(-1)^n x^n}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{4^{n+1}}$

$\boxed{R=4 \quad I:(-4,4)}$

$\sum_{n\geq 0}^{\infty} c_n x^n$ does this by shifting the series

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$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{4^{n+1}} = \sum_{j=2} \frac{(-1)^{j-2} x^j}{4^{j-1}} = \sum_{n=2} \frac{(-1)^{n-2} x^n}{4^{n-1}}$$

$j = n+2$

$j-2 = n$

$\to n+1 = j-1$

Monday, November 4, 2019 7:54 PM

$$\frac{1}{1-D} = \sum_{n=0} D^n \qquad |D|<1$$

Example: Find the power series representation of $f(x)$ and the radius of convergence.

A) $\frac{3x^3}{1-9x^2} = 3x^3 \cdot \frac{1}{1-\boxed{9x^2}} = 3x^3 \cdot \sum_{n=0} (9x^2)^n$

$$= 3x^3 \cdot \sum_{n=0} 9^n x^{2n} = \sum_{n=0} 3 \cdot 3^{2n} x^3 x^{2n}$$

$$= \boxed{\sum_{n=0} 3^{2n+1} x^{2n+3}}$$

$$\boxed{R = \frac{1}{3}}$$

$|9x^2| < 1$

$|x^2| < \frac{1}{9}$

$x^2 < \frac{1}{9}$

$\sqrt{x^2} < \sqrt{\frac{1}{9}}$

$|x| < \frac{1}{3}$

$|x - 0| < \frac{1}{3} = R$

centered

partial fractions

$$\frac{1}{1-\square} = \sum_{n=0} \square^n \quad |\square|<1$$

B) $\frac{x}{x^2-3x+2} = \frac{2}{x-2} + \frac{-1}{x-1}$

multiplied top + bottom by -1

$$= \frac{-2}{2-x} + \frac{1}{1-x}$$

$$= \frac{-2}{2\left[1-\frac{x}{2}\right]} + \frac{1}{1-x}$$

$$= \frac{-1}{1-\boxed{\frac{x}{2}}} + \frac{1}{1-\boxed{x}}$$

$\left|\frac{x}{2}\right|<1$

$|x|<2$

$R=2$

$|x|<1$

$R=1$

$$= -\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n + \sum_{n=0} x^n$$

$$= -\sum_{n=0}\frac{x^n}{2^n} + \sum_{n=0} x^n$$

$$= \sum_{n=0}\frac{-x^n}{2^n} + \sum_{n=0} x^n$$

$$= \sum_{n=0} -\frac{x^n}{2^n} + x^n = \boxed{\sum_{n=0}\left(\frac{-1}{2^n}+1\right)x^n}$$

$R=1$

$\frac{x}{x^2-3\cancel{x}+2}$

$I: (-1,1)$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2^n}+1\right)\left(\frac{1}{2}\right)^n = \frac{1/2}{\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 2}$$

$$\frac{1}{1-\square} = \sum \square^n$$

C) $\dfrac{9}{x^4+81}$ $= \dfrac{9}{81\left[1+\frac{x^4}{81}\right]} = \dfrac{1}{9}\cdot\dfrac{1}{1-\boxed{\frac{-x^4}{81}}}$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{-x^4}{81}\right)^n = \frac{1}{9}\sum_{n=0}\frac{(-1)^n x^{4n}}{9^{2n}}$$

$$= \boxed{\sum_{n=0}\frac{(-1)^n x^{4n}}{9^{2n+1}} \qquad R = 3}$$

$I: (-3, 3)$

$$\left|\frac{-x^4}{81}\right| < 1$$

$$\frac{x^4}{81} < 1$$

$$x^4 < 81$$

$$|x| < \sqrt[4]{81} = 3$$

$$|x| < 3$$

Theorem: If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function defined by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ... = \sum_{n=0}^{\infty} c_n(x-a)^n$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ... = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)dx = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + ... = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$$

The radii of convergence for both $f'(x)$ and $\int f(x)dx$ are both R. The interval of convergence may change.

Pg 8

Monday, November 4, 2019 7:54 PM

$$f = \sum_{n=0}^{\infty} \frac{x^n}{3^n}$$

$$\text{f} = \frac{x^0}{3^0} + \frac{x^1}{3^1} + \frac{x^2}{3^2} + \cdots$$

$$\text{f} = 1 + \frac{x}{3} + \frac{x^2}{3^2} + \cdots$$

$$f' = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n}$$

$$g = \sum_{n=0}^{\infty} \frac{x^{n+2}}{3^n}$$

$$\text{g} = \frac{x^2}{3^0} + \frac{x^3}{3^1} + \frac{x^4}{3^2} + \cdots$$

$$\text{g} = x^2 + \frac{x^3}{3^1} + \frac{x^4}{3^2} + \cdots$$

$$g' = \sum_{n=0}^{\infty} \frac{(n+2)x^{n+1}}{3^n}$$

Pg 9

Monday, November 4, 2019 7:54 PM

Example: Evaluate this integral by using a power series and find the radius of convergence.

$$\int \frac{9}{x^4+81}\,dx$$

$$\boxed{\frac{9}{x^4+81} = \sum_{n=0} \frac{(-1)^n x^{4n}}{9^{2n+1}}} \qquad R = 3$$

$$\int \frac{9}{x^4+81}\,dx = \int \sum_{n=0} \frac{(-1)^n x^{4n}}{9^{2n+1}}\,dx$$

$R = 3$

$$= \sum_{n=0} \int \frac{(-1)^n x^{4n}}{9^{2n+1}}\,dx$$

$$\boxed{\int \frac{9}{x^4+81}\,dx = \sum_{n=0} \frac{(-1)^n x^{4n+1}}{9^{2n+1}(4n+1)} + C \qquad R=3}$$

$$\frac{1}{1-x} = \sum_{n=0} x^n \quad |x|<1 \quad R=1 \quad I:(-1,1)$$

Example: Find a power series representation of $f(x)$ and determine the interval and radius of convergence.

$f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-\boxed{(-x)}} = \sum_{n=0} (-x)^n = \sum_{n=0} (-1)^n x^n$$

$$|-x|<1$$

$$|x|<1 \longrightarrow R=1$$

$$f(x) = \int f'(x)\,dx$$

$$\ln|1+x| = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$\ln|1+x| = C + \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1}$$

$R=1$

$$\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} \cdots$$

need to find the value of C.

pick $x=0$ (where we are centered)

$$\ln(1+0) = C + 0$$

$$\ln(1) = C$$

$$C = 0$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$R=1$

$|x|<1$

Now find the interval of conv.

starts $-1 < x < 1$

Test the end points.

$x=1$ $\quad \displaystyle\sum_{n=0} \frac{(-1)^n 1^{n+1}}{n+1} = \sum_{n=0} \frac{(-1)^n}{n+1}$

$b_n = \frac{1}{n+1}$

dec.

$\lim_{n\to\infty} b_n = 0$

Conv. by AST

$x=-1$ $\quad \displaystyle\sum_{n=0} \frac{(-1)^n (-1)^{n+1}}{n+1} = \sum_{n=0} \frac{\boxed{(-1)^n (-1)^n} (-1)}{n+1}$

$$= \sum \frac{-1}{n+1} = -\sum \frac{1}{n+1}$$

div. by LCT with $\sum \frac{1}{n}$

$$\boxed{\ln(1+x) = \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} \qquad R=1 \quad I: (-1, 1]}$$

Our second building block.

$$\ln(1+\boxed{x}) = \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} \quad R=1 \quad |x|<1$$

Example: Find the power series representation of these functions. determine the radius of convergence.

A) $f(x) = \ln(1-x)$ $= \ln(1+\boxed{(-x)}) = \sum_{n=0} \frac{(-1)^n(-x)^{n+1}}{n+1}$

$|-x|<1$

$|x|<1$

$$= \sum_{n=0} \frac{\boxed{(-1)^n(-1)^{n+1}} x^{n+1}}{n+1} = \sum_{n=0} \frac{-x^{n+1}}{n+1}$$

$R=1$

B) $f(x) = \ln(4+x^2)$ $= \ln\left[4\left(1+\frac{x^2}{4}\right)\right]$

$$= \ln(4) + \ln\left(1+\boxed{\frac{x^2}{4}}\right)$$

$$= \ln(4) + \sum_{n=0} \frac{(-1)^n\left(\frac{x^2}{4}\right)^{n+1}}{n+1}$$

$$= \ln(4) + \sum_{n=0} \frac{(-1)^n}{n+1} \cdot \frac{(x^2)^{n+1}}{4^{n+1}}$$

$$\ln(4+x^2) = \boxed{\ln(4) + \sum_{n=0} \frac{(-1)^n x^{2n+2}}{(n+1)\cdot 4^{n+1}}}$$

$R=2$

$\left|\frac{x^2}{4}\right|<1$

$|x^2|<4$

$x^2<4$

$|x|<\sqrt{4}$

$|x|<2$

$\sqrt{x^2} = |x|$

Example: Find the power series representation of $f(x)$ and determine the radius of convergence.

$f(x) = \arctan(x)$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$|-x^2| < 1$

$|x^2| < 1$

$x^2 < 1$

$|x| < 1 = R$

$$f'(x) = \sum_{n=0} (-1)^n x^{2n} \qquad R = 1$$

$$f(x) = \int f'(x)\,dx = \int \sum_{n=0} (-1)^n x^{2n}\,dx$$

$$\arctan(x) = C + \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad R = 1$$

$$\frac{1x^1}{1} - \frac{x^3}{3} + \cdots$$

We can solve for C. (plug in $x=0$)

$$\arctan(0) = C + 0$$

$$0 = C$$

Building Block

$$\boxed{\arctan(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad R = 1}$$

Pg 13

Monday, November 4, 2019 8:01 PM

$$\boxed{\frac{1}{1-\square} = \sum_{n=0} \square^n \qquad |\square|<1}$$

Example: Find a power series representation of $f(x)$.

$$f(x) = \frac{1}{(1+x)^3} = \frac{1}{2} g''(x)$$

$$g(x) = \frac{1}{1+x} = (1+x)^{-1} = \frac{1}{1-(-x)} = \sum_{n=0} (-1)^n x^n$$

$$= 1 - x + x^2 - x^3 + \dots$$

$$g'(x) = -1(1+x)^{-2}(1) = (-1)(1+x)^{-2} = \sum_{n=1} (-1)^n n x^{n-1}$$

$$= \frac{-1}{(1+x)^2}$$

$$-1 + 2x - 3x^2 + \dots$$

$$g''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3} = \sum_{n=2} (-1)^n n(n-1) x^{n-2}$$

$$f(x) = \frac{1}{2} g''(x) = \frac{1}{2} \sum_{n=2} (-1)^n n(n-1) x^{n-2}$$

$$f(x) = \sum_{n=2} \frac{(-1)^n n(n-1) x^{n-2}}{2}$$

$$\sum_{n=0} \boxed{}$$

Let $j = n-2$

$j+2 = n$

j $\quad j+1 = n-1$

$n+2$

$$\sum_{j=0} \frac{(-1)^{j+2}(j+2)(j+1)x^{j}}{2}$$

$$\boxed{f(x) = \sum_{n=0} \frac{(-1)^{n+2}(n+2)(n+1)x^{n}}{2}}$$

Example: Find a power series representation of $f(x)$.

$$f(x) = \frac{x^3}{(1+2x)^3} = x^3 \cdot \frac{1}{8} g''(x) = \frac{x^3}{8} g''(x)$$

$$g(x) = \frac{1}{1+2x} = (1+2x)^{-1} = \frac{1}{1-(-2x)} = \sum_{n=0} (-1)^n 2^n x^n$$

$$g'(x) = -(1+2x)^{-2}(2) = -2(1+2x)^{-2} = \sum_{n=1} (-1)^n 2^n n x^{n-1}$$

$$g''(x) = 4(1+2x)^{-3}(2) = \frac{8}{(1+2x)^3} = \sum_{n=2}^{\infty} (-1)^n 2^n n(n-1) x^{n-2}$$

$$f(x) = \frac{x^3}{8} g''(x) = \frac{x^3}{2^3} g''(x)$$

$$= \frac{x^3}{2^3} \sum_{n=2} \frac{(-1)^n 2^n n(n-1) x^{n-2}}{2^3}$$

$$= \sum_{n=2} \frac{(-1)^n 2^n n(n-1) x^3 x^{n-2}}{2^3}$$

$$\boxed{f(x) = \sum_{n=2} (-1)^n 2^{n-3} n(n-1) x^{n+1}}$$

$$\arctan(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Example: Use a series to evaluate this integral.

$$\int \arctan(x^3)\ dx$$

$$\arctan(x^3) = \sum_{n=0} \frac{(-1)^n (x^3)^{2n+1}}{2n+1}$$

$$= \sum_{n=0} \frac{(-1)^n x^{6n+3}}{2n+1}$$

$$\int \arctan(x^3) dx = \int \sum_{n=0} \frac{(-1)^n x^{6n+3}}{2n+1} dx$$

$$\boxed{\int \arctan(x^3) dx = C + \sum_{n=0} \frac{(-1)^n x^{6n+4}}{(2n+1)(6n+4)}}$$