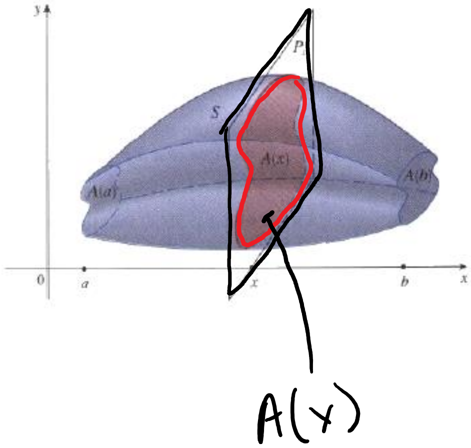
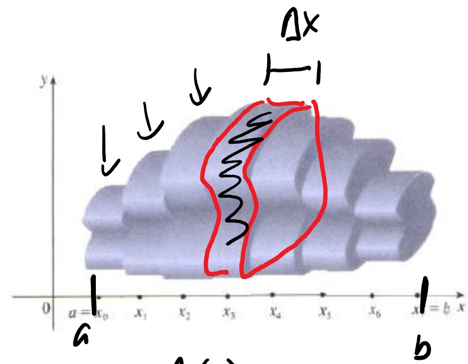
Section 6.2: Volume

Let S be a sold that lies between the planes P_a and P_b. Assume that cross sections of the solid is given by A and are perpendicular to the x-axis.



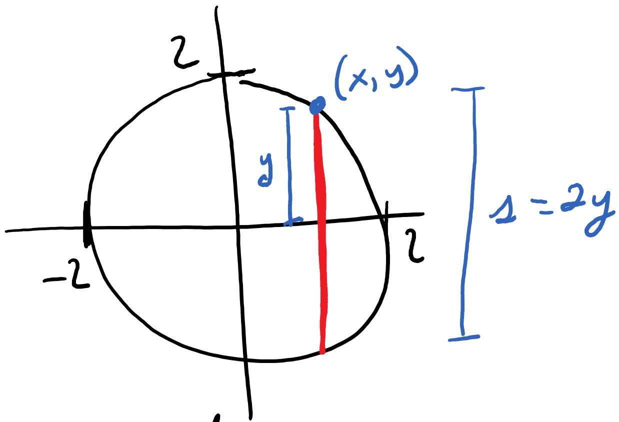


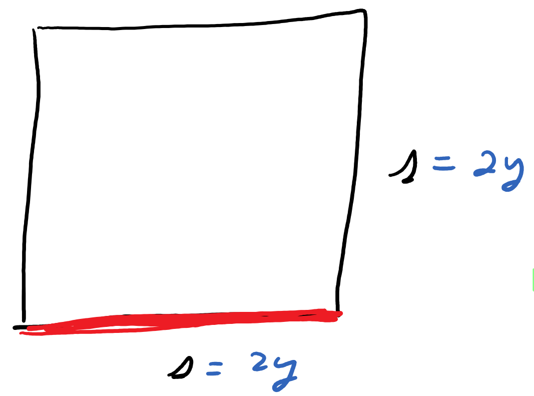
$$V_i = A(x)\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x)\Delta x$$

$$V = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum A(x)\,\Delta x = \int_a^b A(x)\,dx$$

Example: The solid, S, has a base that is a circular disk with radius 2. Find the volume of the the solid if parallel cross sections taken perpendicular to the base are squares.





dx Integral

$$x^2+y^2=2^2$$

$$\boxed{y^2=4-x^2}$$

$$A = s^2 = (2y)^2 = 4y^2$$

$$A = 4(4-x^2)$$

$$= 16-4x^2$$

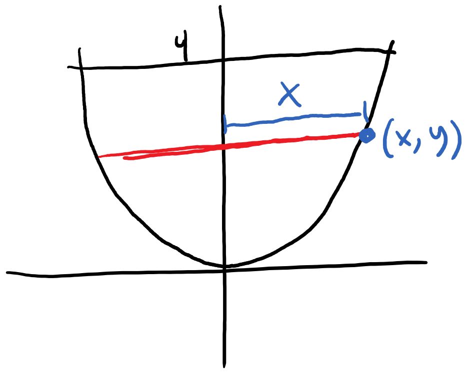
$$V = \int_{-2}^{2} 16-4x^2\,dx = 2\int_0^2 16-4x^2\,dx$$

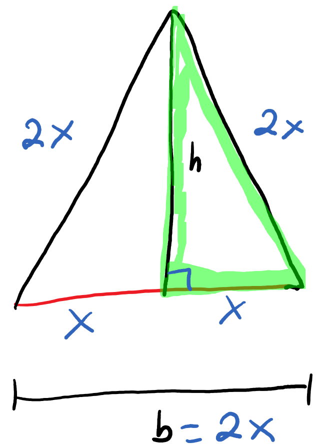
$$= 2\left(16x - \frac{4x^3}{3}\right)\Big|_0^2 = 2\left(32-\frac{32}{3}\right) - 2(0)$$

$$= \frac{128}{3}$$

Example: The solid, S, has a base that is bounded by the equations: $y = x^2$ and $y = 4$. Find the volume of the solid if parallel cross sections are equilateral triangles that are perpendicular to the y-axis

dy Integral





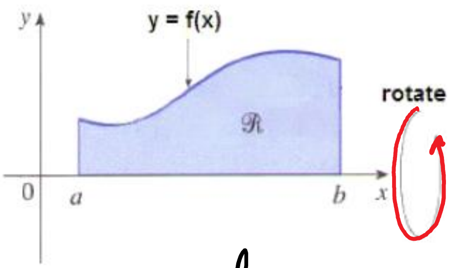
$$x^2 + h^2 = (2x)^2$$
$$x^2 + h^2 = 4x^2$$
$$h^2 = 3x^2$$
$$h = x\sqrt{3}$$

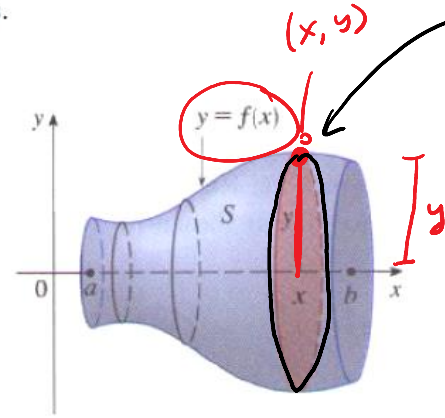
$$A = \frac{1}{2} b h = \frac{1}{2}(2x) \cdot x\sqrt{3}$$
$$A = x^2\sqrt{3}$$
$$A = y\sqrt{3}$$

$$V = \int_0^4 y\sqrt{3}\, dy = \frac{\sqrt{3}\, y^2}{2} = \frac{16\sqrt{3}}{2} - 0 = 8\sqrt{3}$$

Page 4: disk method

Now lets consider rotating a region bounded between the x-axis and the function $f(x)$ from $x = a$ to $x = b$ around the x-axis.





disk.

$A = \pi r^2$

$A = \pi y^2$

$A = \pi (f(x))^2$

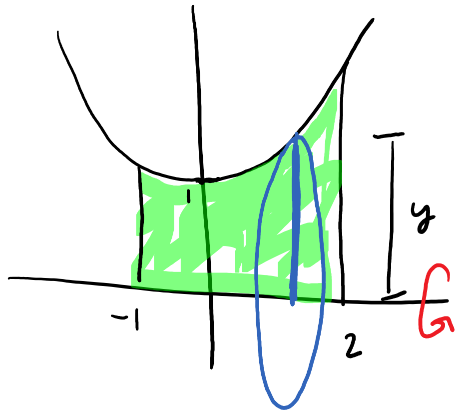
dx Integral

$$V = \int_a^b \pi (f(x))^2 \, dx$$

Example: Find the volume of the solid obtained by rotating the region bounded by the following around the x-axis.

$y = x^2 + 1$
x-axis
$x = -1$
$x = 2$

dx Integral



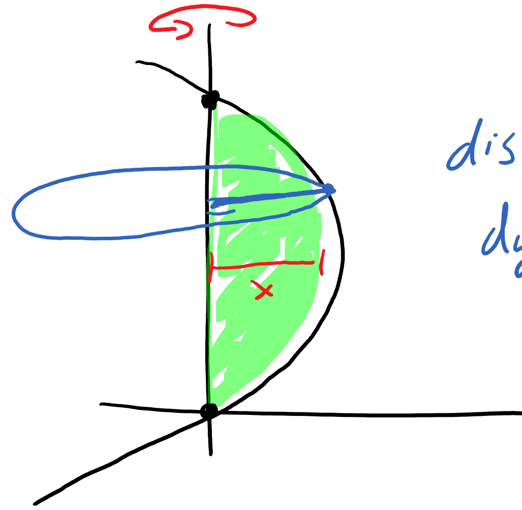
$A = \pi y^2$

$A = \pi (x^2+1)^2$

$$V = \int_{-1}^{2} \pi (x^2+1)^2 \, dx = \pi \int_{-1}^{2} x^4 + 2x^2 + 1 \, dx = \dots = 15.6\pi$$

Example: Find the volume of the solid obtained by rotating the region bounded by the given curves around the y-axis.

$x = 4y - y^2 = y(4-y)$
$x = 0$



disk.

dy Integral

$0 = y(4-y)$

$y = 0 \quad y = 4$

$A = \pi x^2$

$A = \pi (4y - y^2)^2$

$$V = \int_0^4 \pi (4y - y^2)^2 \, dy = \cdots = \frac{512\pi}{15}$$

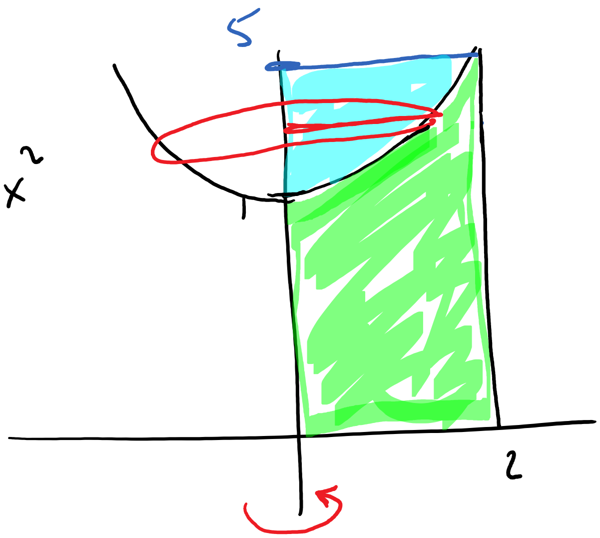
Example: Find the volume of the solid obtained by rotating the region bounded by the given curves around the y-axis.

$y = x^2 + 1$
$y = 0$
$x = 0$
$x = 2$

$y - 1 = x^2$

$r = x$

$A = \pi x^2$
$= \pi (y-1)$



Cylinder (Blue + Green)

$V = \pi r^2 h$
$V = \pi (2)^2 (5)$
$V = 20\pi$

Blue Region

$$V = \int_1^5 \pi (y-1)\, dy$$

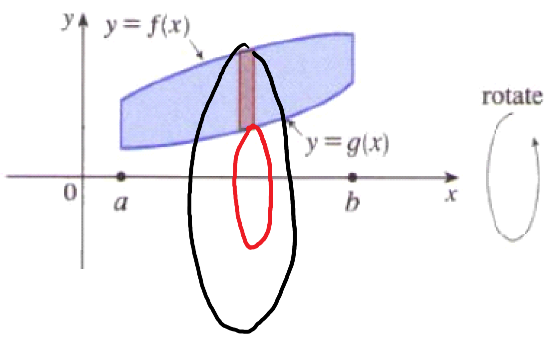
$V = \dots = 8\pi$

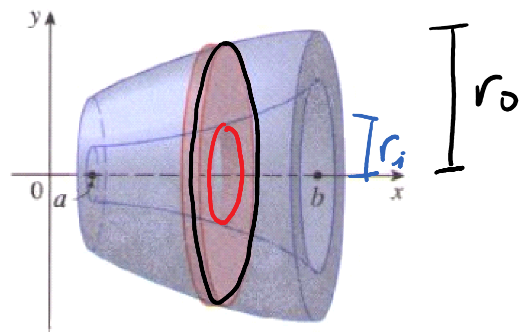
Answer $= 20\pi - 8\pi = \underline{\underline{12\pi}}$

Page 8: washer method

Now lets consider rotating a region bounded between the function $f(x)$ and $g(x)$ from $x = a$ to $x = b$ around the x-axis.

washer.





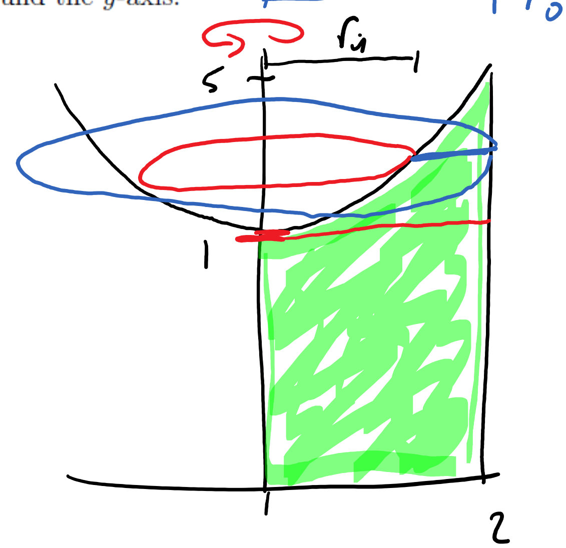
$$A = \pi r_o^2 - \pi r_i^2 = \pi\left[r_o^2 - r_i^2 \right]$$

$$V = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx$$

Example: Find the volume of the solid obtained by rotating the region bounded by the given curves around the y-axis.

$y = x^2 + 1$
$y = 0$
$x = 0$
$x = 2$

dy Integral



Top

$r_o = 2$

$r_i = x \to (r_i)^2 = x^2$
$= 1 - y$

Bottom cylinder

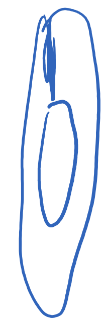
$V = \pi(2)^2(1)$
$= 4\pi$

Top

$$V = \int_1^5 \pi\left[2^2 - (1-y)\right] dy = \cdots = 8\pi$$

Answer $= 8\pi + 4\pi = 12\pi$

dx Integral

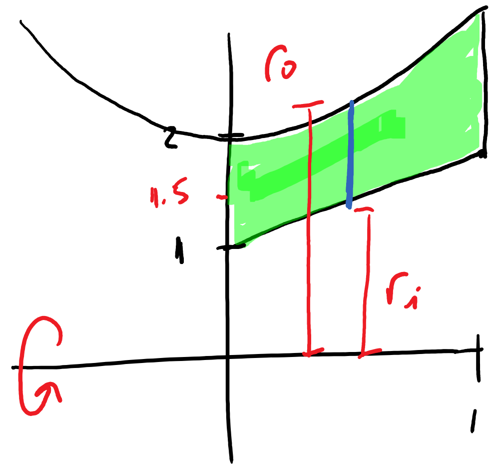


Example: Set up the integral(s) that would give the volume of the solid obtained by rotating the region bounded by the given curves around x-axis.

$y = x^2 + 2$
$2y - x = 2$
$x = 0$
$x = 1$

$2y = x + 2$

$y = \frac{1}{2}x + 1$



$r_o = y_{parabola}$

$r_o = x^2 + 2$

$r_i = y_{Line}$

$r_i = \frac{1}{2}x + 1$

$$V = \int_0^1 \pi \left[(x^2+2)^2 - \left(\frac{1}{2}x+1\right)^2 \right] dx$$

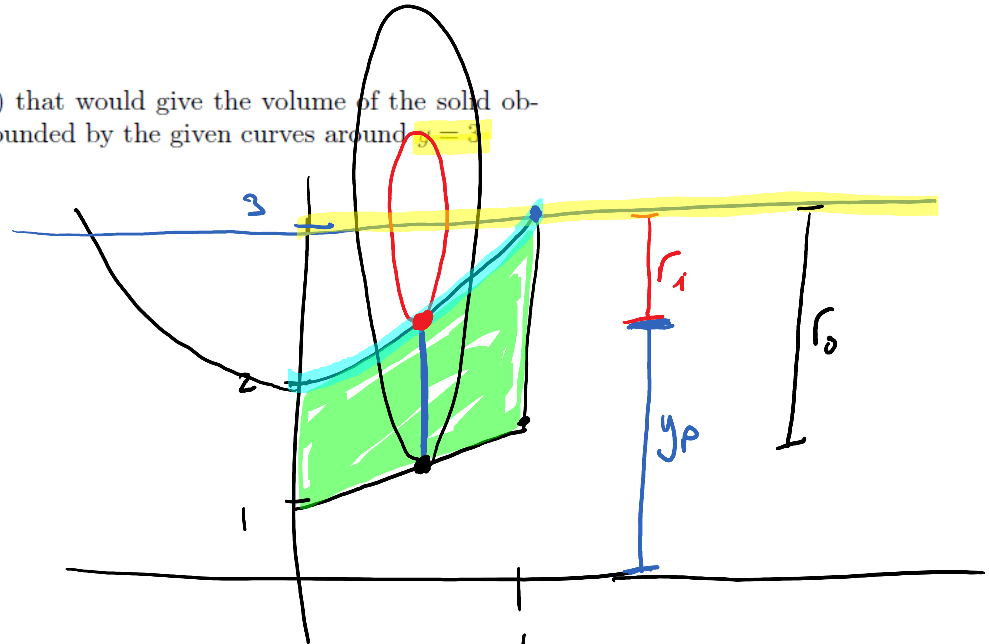
$$= \cdots = \frac{79\pi}{20}$$

Example: Set up the integral(s) that would give the volume of the solid obtained by rotating the region bounded by the given curves around $y = 3$

$y = x^2 + 2$
$2y - x = 2$
$x = 0$
$x = 1$

$y = \frac{1}{2}x + 1$

dx Integral



$r_i + y_p = 3 \longrightarrow r_i = 3 - y_p = \underset{\text{TOP}}{3} - \underset{\text{Bottom}}{(x^2+2)} = 3 - x^2 - 2 = 1 - x^2$

$r_o = 3 - y_L = 3 - \left(\frac{1}{2}x + 1\right) = 3 - \frac{1}{2}x - 1 = 2 - \frac{1}{2}x$

$$V = \int_0^1 \pi \left[\left(2 - \frac{1}{2}x\right)^2 - \left(1 - x^2\right)^2\right] dx = \cdots = \frac{51}{20}\pi$$

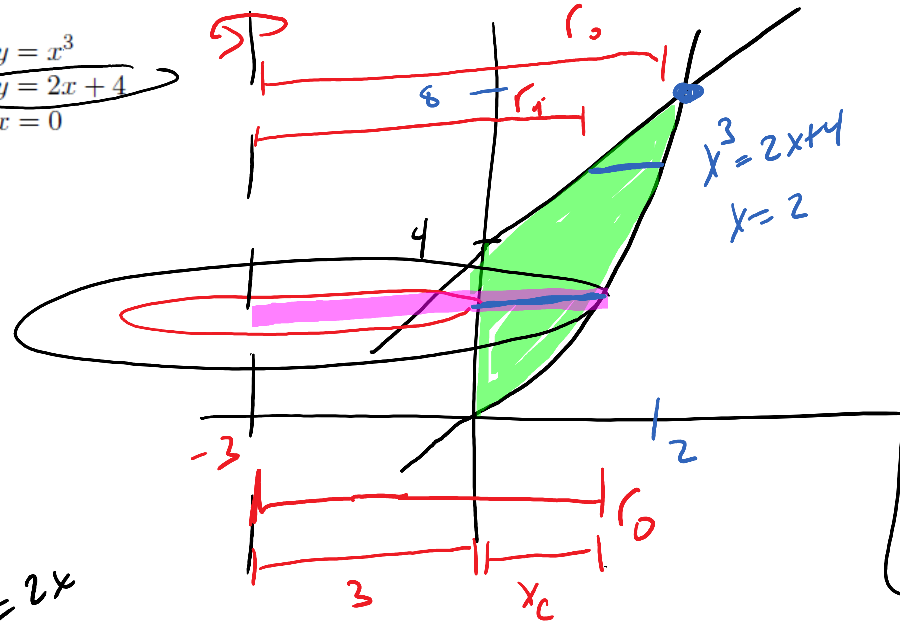
dy Integral $\qquad y = x^3 \to x = y^{1/3}$

Example: Set up the integral(s) that would give the volume of the solid obtained by rotating the region bounded by the given curves around $x = -3$.

$y = x^3$

$y = 2x + 4$

$x = 0$



Bottom

$r_o = x_c + 3$

$r_o = y^{1/3} + 3$

Right − Left

$r_o = x_c - (-3)$

$r_o = y^{1/3} + 3$

$r_i = 3$

$y - 4 = 2x$

$x = \frac{1}{2}y - 2$

Top

$r_o = y^{1/3} + 3$

$r_i = x_L - (-3)$

$r_i = \frac{1}{2}y - 2 + 3$

$r_i = \frac{1}{2}y + 1$

$$V = \int_4^8 \pi\left[\left(y^{1/3}+3\right)^2 - \left(\frac{1}{2}y+1\right)^2\right]dy + \int_0^4 \pi\left[\left(y^{1/3}+3\right)^2 - 3^2\right]dy$$

$$= \cdots = \frac{928\pi}{15}$$