Section 7.4: Integration of Rational Functions by Partial Fractions

$$\frac{3}{x+2}+\frac{4}{x+5}=\frac{3(x+5)+4(x+2)}{(x+2)(x+5)}=\frac{7x+23}{x^2+7x+10}$$

$$\int \frac{7x+23}{x^2+7x+10}\,dx=\int \frac{3}{x+2}+\frac{4}{x+5}\,dx= \quad 3\ln|x+2|+4\ln|x+5|+C$$

A rational function is a function of the form $\frac{P(x)}{Q(x)}$ where both $P(x)$ and $Q(x)$ are polynomials. The degree of a polynomial is the highest power of the variable.

<u>**NOTE:**</u> To integrate a rational function, $\frac{P(x)}{Q(x)}$, with the partial fraction method, you <u>MUST HAVE</u> the degree $P(x) <$ degree $Q(x)$. If this is not the case then use long division(or some other method) to find $J(x)$ and $K(x)$ so that

$$\frac{P(x)}{Q(x)}=J(x)+\frac{K(x)}{Q(x)}$$

Method of Integration by Partial Fractions:

0) Do long division(or other algebra manipulation) if degree $P(x) \geq$ degree $Q(x)$.
1) Factor the denominator completely
2) Decompose the fraction
3) Solve for the constants in the decomposition
4) Integrate the new fractions

Example: Compute these Integrals.

A) $\int \frac{x^3+2x^2-5}{x+1}\,dx \quad = \int x^2+x-1-\frac{4}{x+1}\,dx$

$$\begin{array}{r|l} & x^2+x-1 \\ \hline x+1 & x^3+2x^2+0x-5 \\ & -(x^3+x^2) \\ \hline & x^2+0x \\ & -(x^2+x) \\ \hline & -x-5 \\ & -(-x-1) \\ \hline & -4 \end{array}$$

$$= \frac{x^3}{3}+\frac{x^2}{2}-x-4\ln|x+1|+C$$

B) $\int \frac{x}{x+5}\,dx = \int \frac{u-5}{u}\,du = \int \frac{u}{u} - \frac{5}{u}\,du = \int 1 - \frac{5}{u}\,du$

$u = x+5$

$du = dx$

$u-5 = x$

$= \cancel{u} - 5\ln|u| + C$

$= x+5 - 5\ln|x+5| + C$

$\int \frac{x}{x+5}\,dx = \int 1 - \frac{5}{x+5} = x - 5\ln|x+5| + C$

$$\begin{array}{r} 1 \\ x+5 \,\overline{)\, x + 0} \\ -(x+5) \\ \hline -5 \end{array}$$

$\int \frac{x}{x+5}\,dx = \int \frac{x+5}{x+5} - \frac{5}{x+5}\,dx$

$= \int 1 - \frac{5}{x+5}\,dx = x - 5\ln|x+5| + C$

C) $\int \frac{x^3+3x-5}{x^2+1}\,dx$

$$\begin{array}{r} x \\ x^2+0x+1 \overline{)\, x^3+0x^2+3x-5} \\ -(x^3+0x^2+x) \downarrow \\ \hline 2x-5 \end{array}$$

$$\int \frac{x^3+3x-5}{x^2+1}\,dx = \int x + \frac{2x-5}{x^2+1}\,dx$$

$$= \int x + \frac{2x}{x^2+1} - \frac{5}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + \ln|x^2+1| - 5\arctan(x) + C$$

Example: Write out the partial fraction decomposition. Do not determine the numerical values of the coefficients.

A) $\dfrac{-3x+20}{x^3+3x^2-10x} = \dfrac{-3x+20}{x(x^3+3x-10)} = \dfrac{-3x+20}{x(x-2)(x+5)} = \boxed{\dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+5}}$

B) $\dfrac{x-3}{x(x+1)^3(x^2+5)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} + \dfrac{d}{(x+1)^3} + \dfrac{fx+g}{x^2+5}$

multiplicity of that term

C) $\dfrac{x^2+2}{x^3(x^2-9)(x^2+16)^2} = \dfrac{x^2+2}{x^3(x+3)(x-3)(x^2+16)^2}$

$= \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{d}{x+3} + \dfrac{e}{x-3} + \dfrac{fx+g}{x^2+16} + \dfrac{hx+m}{(x^2+16)^2}$

Example: Compute these integrals.

A) $\int \frac{-3x+20}{x^3+3x^2-10x}\,dx$

$$\frac{-3x+20}{\underset{x(x-2)(x+5)}{x^3+3x^2-10x}} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

$$\frac{-3x+20}{x(x-2)(x+5)} = \frac{A\,(x-2)(x+5)}{x\,(x-2)(x+5)} + \frac{B\,x\,(x+5)}{(x-2)\,x\,(x+5)} + \frac{C\,x\,(x-2)}{(x+5)\,x\,(x-2)}$$

$$\frac{-3x+20}{x(x-2)(x+5)} = \frac{A(x-2)(x+5) + Bx(x+5) + Cx(x-2)}{x(x-2)(x+5)}$$

$$-3x+20 = A(x-2)(x+5) + Bx(x+5) + Cx(x-2)$$

$$-3x+20 = A(x^2+3x-10) + B(x^2+5x) + C(x^2-2x)$$

$$-3x+20 = Ax^2 + 3Ax - 10A + Bx^2 + 5Bx + Cx^2 - 2Cx$$

$$0x^2 - 3x + 20 = (A+B+C)x^2 + (3A+5B-2C)x - 10A$$

x^2] $0 = A+B+C$

x] $-3 = 3A+5B-2C$

$20 = -10A$

$A = -2$

x | $-3 = 3A + 5B - 2C$

Const | $20 = -10A$

$-3 = -6 + 5B - 2C$

$0 = -2 + B + C$

$3 = 5B - 2C$

$2 = B + C$

$3 = 5B - 2(2 - B)$

$2 - B = C$

$3 = 5B - 4 + 2B$

$7 = 7B$

$C = 2 - 1 = 1$

$B = 1$

$$\text{Integral} = \int \frac{-2}{x} + \frac{1}{x-2} + \frac{1}{x+5}\,dx$$

$$= -2\ln|x| + \ln|x-2| + \ln|x+5| + C$$

Example: Compute these integrals.

A) $\int \frac{-3x+20}{x^3+3x^2-10x}\,dx$

$$\boxed{\frac{-3x+20}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}}$$

$$\frac{-3x+20}{x(x-2)(x+5)} = \frac{A(x-2)(x+5)}{x(x-2)(x+5)} + \frac{B\,x(x+5)}{(x-2)\,x(x+5)} + \frac{C\,x(x-2)}{(x+5)\,x(x-2)}$$

$$\boxed{\frac{-3x+20}{x(x-2)(x+5)} = \frac{A(x-2)(x+5)+Bx(x+5)+Cx(x-2)}{x(x-2)(x+5)}}$$

$$-3x+20 = A(x-2)(x+5) + Bx(x+5) + Cx(x-2)$$

if $x=0$

$20 = A(-2)(5)$

$20 = -10A$

$A = -2$

if $x=2$

$-6+20 = B\,2(7)$

$14 = 14B$

$B = 1$

if $x=-5$

$15+20 = C(-5)(-7)$

$35 = 35C$

$C = 1$

dx

Integral $= \int \frac{-2}{x} + \frac{1}{x-2} + \frac{1}{x+5}\, dx$

$$= -2\ln|x| + \ln|x-2| + \ln|x+5| + C$$

B) $\int \frac{x+2}{x^3+2x}\,dx$

$$\frac{x+2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$x+2 = A(x^2+2) + (Bx+C)x$$

$$x+2 = Ax^2 + 2A + Bx^2 + Cx$$

if $x=0$

$2 = A(2) + 0$

$2A = 2$

$A = 1$

x^2 | $0 = A + B$

x | $1 = C$

Const | $2 = 2A$

$B = -1$

$$\int \frac{x+2}{x(x^2+2)}\,dx = \int \frac{1}{x} + \frac{-x+1}{x^2+2}\,dx$$

$$= \int \frac{1}{x} - \frac{x}{x^2+2} + \frac{1}{x^2+2}\,dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+2}\,dx + \int \frac{1}{\underset{x^2+A^2}{x^2+2}}\,dx$$

$A = \sqrt{2}$

$u = x^2+2$

$$= \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$$

C) $\int \frac{15x+5}{(x+2)^2(x^2+1)}\,dx$

$$\frac{15x+5}{(x+2)^2(x^2+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+d}{x^2+1}$$

$$15x+5 = A(x+2)(x^2+1) + B(x^2+1) + (Cx+d)\underbrace{(x+2)^2}_{x^2+4x+4}$$

Let $x=-2$

$-25 = B(5)$

$-5 = B$

$$15x+5 = A(x^3+2x^2+x+2) + B(x^2+1) + C(x^3+4x^2+4x) + d(x^2+4x+4)$$

x^3 | $0 = A + C$

x^2 | $\boxed{0 = 2A + B + 4C + d}$

x | $15 = A + 4C + 4d$

const | $5 = 2A + B + 4d$

$B = -5$

$C = -A$

$0 = 2A - 5 - 4A + d$

$0 = -5 - 2A + d$

$d = 5 + 2A$

$5 = 2A - 5 + 4(5+2A)$

$5 = 2A - 5 + 20 + 8A$

$5 = 10A + 15$

$-10 = 10A$

$A = -1$ $\qquad C = 1$

$d = 5 + 2H$

$A = -1 \qquad C = 1$

$d = 5 - 2$

$d = 3$

$$\int \frac{-1}{x+2} + \frac{-5}{(x+2)^2} + \frac{1x+3}{x^2+1}\, dx$$

$$= \int \frac{-1}{x+2}\, dx + \int \frac{-5}{(x+2)^2}\, dx + \int \frac{x}{x^2+1}\, dx + \int \frac{3}{x^2+1}\, dx$$

$u = x+2 \qquad u = x^2+1$

$$= -\ln|x+2| + \frac{5}{x+2} + \frac{1}{2}\ln|x^2+1| + 3\arctan(x) + C$$