Math 152 Exam 2 Review

Solutions and questions can be found at the link:
https://www.math.tamu.edu/~kahlig/152WIR.html

The following is a collection of questions to review the topics for the second exam. This is not intended to represent an actual exam nor does it have every type of problem seen int he homework.
These questions cover sections 7.3, 7.4, 7.8, 11.1, 11.2, and 11.3.

1. Determine if the series converges. $\sum_{n=1}^{\infty} \frac{3}{n^2+4}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{n^2+4} = 0$$

$$f(x) = \frac{3}{x^2+4}$$

$$f'(x) = \frac{0 - 3(2x)}{(x^2+4)^2} = \frac{-6x}{(x^2+4)^2}$$

$f'(x) < 0$ for $x > 0$

$f(x)$ is dec

continuous ✓
positive ✓
dec ✓

$$\int \frac{1}{x^2+A^2}\,dx = \frac{1}{A}\arctan\left(\frac{x}{A}\right)$$

$$\int_1^{\infty} \frac{3}{x^2+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{3}{x^2+4}\,dx$$

$$= \lim_{t\to\infty} \frac{3}{2}\arctan\left(\frac{x}{2}\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{3}{2}\arctan\left(\frac{t}{2}\right) - \frac{3}{2}\arctan\left(\frac{1}{2}\right)\right)$$

$$= \frac{3}{2}\cdot\frac{\pi}{2} - \frac{3}{2}\arctan\left(\frac{1}{2}\right)$$

The Integral converges.

by The Integral test the series converges

$$\int_1^\infty \frac{3}{x^2+4}\,dx$$

$x^2+4 > x^2$

$\frac{1}{x^2+4} < \frac{1}{x^2}$

$\frac{3}{x^2+4} < \frac{3}{x^2}$

$\int_1^\infty \frac{3}{x^2}\,dx$ p-integral $p=2$ conv.

By comparison we get $\int_1^\infty \frac{3}{x^2+4}\,dx$ conv.

By The Integral test the series conv.

2. Consider the series $\sum_{n=1}^{\infty} 2ne^{-n^2}$

(a) Show the series converges.

(b) Find an upperbound on the error when using s_4 to approximate the summation.

$f(x) = 2x e^{-x^2}$

Cont. ✓
positive. ✓
dec

$f' = 2e^{-x^2} + 2x \cdot (-2x) e^{-x^2}$

$= 2e^{-x^2} - 4x^2 e^{-x^2}$

$f'(x) = (2 - 4x^2) e^{-x^2}$

$f'(x) < 0$ for $x > 0$

$f(x)$ dec.

$$\int_1^{\infty} 2x e^{-x^2} dx = \lim_{t \to \infty} \int_1^t 2x e^{-x^2} dx$$

$e^{-t^2} = \frac{1}{e^{t^2}}$

$\int 2x e^{-x^2} dx = \int -e^u du$

$= -e^u$

$= -e^{-x^2}$

$u = -x^2$

$du = -2x dx$

$-du = 2x dx$

$= \lim_{t \to \infty} -e^{-x^2} \Big|_1^t$

$= \lim_{t \to \infty} \left(-e^{-t^2} - -e^{-1^2} \right)$

$= 0 + e^{-1} = \frac{1}{e}$

The Integral converges so by the Integral test the

Series Conv.

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approximat S_4 then R_4 Remainder

S_n R_n

$$R_n \le \int_n^\infty f(x)\,dx$$ max error.

$$R_4 \le \int_4^\infty 2x e^{-x^2}\,dx = e^{-4^2} = e^{-16}$$

from the work above.

smallest # of terms

3. How many terms of the series do we need to add in order to find the sum so that the error is $< \frac{1}{20}$?

$\sum_{n=1}^{\infty} \frac{2}{n^3}$

$$R_n \le \int_n^\infty \frac{2}{x^3}\,dx < \frac{1}{20}$$

$$\lim_{t\to\infty} \int_n^t 2x^{-3}\,dx = \lim_{t\to\infty} \left.\frac{2x^{-2}}{-2}\right|_n^t = \lim_{t\to\infty} \left. -x^{-2} \right|_n^t$$

$$= \lim_{t\to\infty} \left.\frac{-1}{x^2}\right|_n^t = \lim_{t\to\infty}\left(-\frac{1}{t^2} - \frac{-1}{n^2}\right)$$

$$= \frac{1}{n^2}$$

$$\frac{1}{n^2} < \frac{1}{20}$$

$$20 < n^2$$

$n = 5$ terms

4. Give the form of the partial-fraction decomposition for the rational function

$$\frac{x+7}{(x-1)\underbrace{(x^2-1)}(x^2+4)} = \frac{A}{X-1} + \frac{B}{(X-1)^2} + \frac{C}{X+1} + \frac{dx+e}{x^2+4}$$

$(X-1)(X-1)(X+1)(x^2+4)$

$(X-1)^2(X+1)(X^2+4)$

5. Compute $\int \frac{3x^2+4x+3}{x^2+1}\,dx$ $= \int 3 + \underline{\frac{4x}{x^2+1}}\,dx$

$= \underline{3x + 2\ln|x^2+1| + C}$

$$\begin{array}{r} 3 \\ x^2+0x+1 \overline{\smash{)}\,3x^2+4x+3} \\ -(3x^2+0x+3) \\ \hline 4x \end{array}$$

$\int \frac{4x}{x^2+1}\,dx = \int \frac{2}{u}\,du$

$u = x^2+1$

$du = 2x\,dx$

$= 2\ln|u|$

$= 2\ln|x^2+1|$

6. Compute $\int \frac{1}{\sqrt{x^2+4x}}\,dx$ $= \int \frac{1}{\sqrt{(x+2)^2-4}}\,dx$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$x^2+4x + 2^2 - 2^2$

$(x+2)^2 - 4$

$4\tan^2\theta = 4\sec^2\theta - 4$

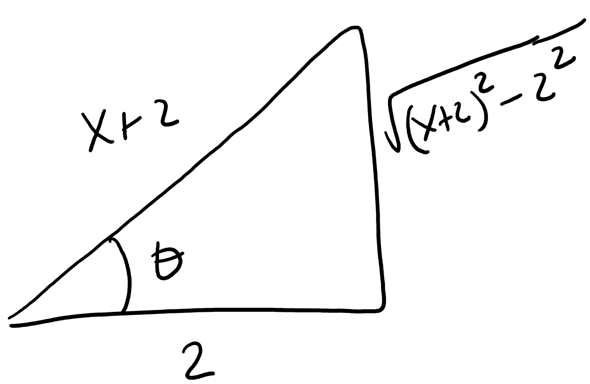
need $(x+2)^2 = 4\sec^2\theta$

let $\boxed{x+2 = 2\sec\theta}$

$x = -2 + 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$\frac{x+2}{2} = \sec\theta$



$$\int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta-4}}\,d\theta$$

$$= \int \frac{2\sec\theta\tan\theta}{\sqrt{4\tan^2\theta}}\,d\theta$$

$$= \int \frac{2\sec\theta\tan\theta}{2\tan\theta}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x+2}{2} + \frac{\sqrt{(x+2)^2-4}}{2}\right| + C$$

7. Compute $\int \frac{dx}{(x^2+4)^2}$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$4\sec^2\theta = 4 + 4\tan^2\theta$

need $x^2 = 4\tan^2\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{2\sec^2\theta}{(4\tan^2\theta + 4)^2}\, d\theta$$

$$= \int \frac{2\sec^2\theta}{(4\sec^2\theta)^2}\, d\theta$$

$$= \int \frac{2\sec^2\theta}{16\sec^4\theta}\, d\theta = \int \frac{1}{8}\frac{1}{\sec^2\theta}\, d\theta$$

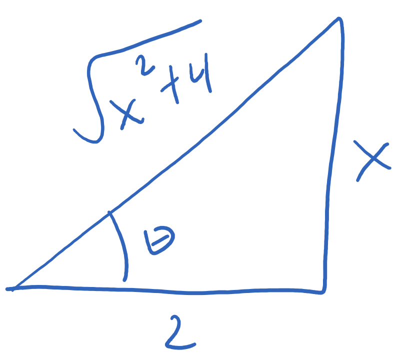
$$= \int \frac{1}{8}\cos^2\theta\, d\theta$$

$$= \frac{1}{8}\cdot\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$



$$= \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right]$$

$$= \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{x}{\quad}\cdot\frac{2}{\quad}\right) + C$$

$$= \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{x}{\sqrt{x^2+4}} \cdot \frac{2}{\sqrt{x^2+4}}\right) + C$$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

8. Compute $\int \frac{2x^3+11x^2+18}{x^2(x^2+9)}\,dx$

$$\frac{2x^3+11x^2+18}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+d}{x^2+9}$$

$$2x^3+11x^2+18 = Ax(x^2+9) + B(x^2+9) + (Cx+d)x^2$$

$$2x^3+11x^2+18 = Ax^3+9Ax + Bx^2+9B + Cx^3+dx^2$$

x^3: $2 = A + C$

x^2: $11 = B + d$

x: $0 = 9A \longrightarrow A = 0 \longrightarrow C = 2$

const: $18 = 9B \longrightarrow B = 2 \longrightarrow d = 9$

$$\int \frac{2}{x^2} + \frac{2x+9}{x^2+9}\,dx = \int \frac{2}{x^2} + \underbrace{\frac{2x}{x^2+9}}_{u=x^2+9} + \frac{9}{x^2+9}\,dx$$

$$= -\frac{2}{x} + \ln|x^2+9| + \frac{9}{3}\arctan\left(\frac{x}{3}\right) + C$$

9. Compute $\int_0^4 \frac{1}{(x-4)^4}\,dx$

$0 \qquad t \to 4$

$\frac{1}{(x-4)^4}$ not cont at $x=4$

vertical asym

$$= \lim_{t \to 4^-} \int_0^t (x-4)^{-4}\,dx = \lim_{t\to 4^-} \left. \frac{-1}{3(x-4)^3} \right|_0^t$$

$$\int (x-4)^{-4}\,dx = \int u^{-4}\,du$$

$u = x-4$

$du = dx$

$$= \frac{u^{-3}}{-3}$$

$$= -\frac{1}{3u^3}$$

$$= \frac{-1}{3(x-4)^3}$$

$$= \lim_{t\to 4^-} \left(\underbrace{\frac{-1}{3(t-4)^3}} - \frac{-1}{3(-4)^3} \right)$$

$$= +\infty$$

diverges (to $+\infty$)

$$\frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

10. Compute $\displaystyle\int_2^\infty \frac{dx}{x(2x+1)} = \int_2^\infty \frac{1}{x} - \frac{2}{2x+1}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x} - \frac{2}{2x+1}\,dx$

$$= \lim_{t\to\infty} \Big(\ln|x| - \ln|2x+1| \Big)\Big|_2^t$$

$$= \lim_{t\to\infty} \Big(\ln(t) - \ln(2t+1) \Big) - \Big[\ln(2) - \ln(5) \Big]$$

$$= \lim_{t\to\infty} \left(\ln\left(\frac{t}{2t+1}\right) - \ln\left(\frac{2}{5}\right) \right)$$

$$= \underbrace{\ln\left(\frac{1}{2}\right) - \ln\left(\frac{2}{5}\right)}$$

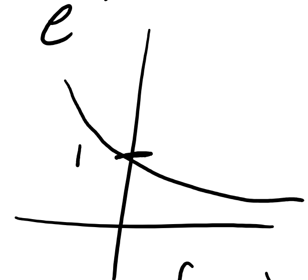
The integral converges to

$$\boxed{\lim_{t\to\infty} \frac{t}{2t+1} \overset{L'H}{=} \lim_{t\to\infty} \frac{1}{2} = \frac{1}{2}}$$

11. Use the Comparison Test for Improper Integrals to decide on the convergence/divergence of each of the following improper integrals.

(a) $\displaystyle\int_1^{\infty} \frac{2+e^{-x}}{x}\,dx$

e^{-x}



for $x > 0$

$e^{-x} < 1$

$$2 < 2 + e^{-x} < 3$$

$$\underbrace{\frac{2}{x}} < \underbrace{\frac{2+e^{-x}}{x}} < \frac{3}{x}$$

$$\boxed{\int_1^{\infty} \frac{3}{x}\,dx \quad p \text{ int. } p=1 \quad \text{div.}}$$

$\displaystyle\int_1^{\infty} \frac{2}{x}\,dx$ div. p-integral $p=1$

by comparison $\displaystyle\int_1^{\infty} \frac{2+e^{-x}}{x}\,dx$ will div.

p-integral

$\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ conv $p>1$

div. $p \le 1$

$\displaystyle\int^{\infty} \frac{4}{x^p}\,dx$

$$\int_3^1 \frac{1}{x^p}\,dx$$

(b) $\int_1^{\infty} \frac{1}{\sqrt{x}+x\sqrt{x}}\,dx$

$\sqrt{x} + x\sqrt{x} > \sqrt{x}$

$\frac{1}{\sqrt{x}+x\sqrt{x}} < \frac{1}{\sqrt{x}}$

$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ p-int. $p=\frac{1}{2}$ div.

no help

$\sqrt{x} + x\sqrt{x} > x\sqrt{x}$

$\frac{1}{\sqrt{x}+x\sqrt{x}} < \frac{1}{x\sqrt{x}} = \frac{1}{x^{1.5}}$

$\int_1^{\infty} \frac{1}{x^{1.5}}\,dx$ p-int. $p=1.5>1$ conv.

$(-1)^n$ $\underbrace{(-1)^{n+1} = (-1)^{n-1}}$

12. Find a formula for the general term, a_n, of the sequence assuming that the pattern of the first few terms continues. Give the formula so the first term is a_1.

$n=1$ $n=2$ $n=3$ $n=4$

$$\left\{\frac{1}{2}, \frac{-4}{5}, \frac{9}{8}, \frac{-16}{11}, \ldots\right\}$$

3 3 3

$$\boxed{a_n = \frac{(-1)^{n+1} n^2}{3n-1}}$$

Term value

(x, y)
$(1, 2)$
$(2, 5)$
$(3, 8)$
$(4, 11)$

$m = 3$

$y - 2 = 3(x-1)$

$y - 2 = 3x - 3$

$y = 3x - 1$

13. Determine whether each sequence converges or diverges. If it converges, find the value.

(a) $a_n = \arctan\left(\frac{n^2}{n+5}\right)$

$$\lim_{n\to\infty} \arctan\left(\underbrace{\frac{n^2}{n+5}}\right) = \frac{\pi}{2}$$

conv.

$$\lim_{n\to\infty} \frac{n^2}{n+5} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{1} = \infty$$

(b) $a_n = \frac{(-1)^n n^2}{2n^2+5}$

$b_n = \frac{n^2}{2n^2+5}$

as $n \to \infty$ $b_n \to \frac{1}{2}$

Thus a_n div.

$a_n = (-1)^n b_n$

if $b_n \to 0$ as $n \to \infty$ then $a_n \to 0$

b_n do not conv. to 0 then a_n div.

14. Assume that the sequence defined below is bounded and is decreasing. Determine if the sequence is convergent or divergent. Give the value the sequence will converge to if it converges.

$$a_1 = 2, \qquad a_{n+1} = \frac{6}{7 - a_n}$$

$\lim_{n \to \infty} a_n = L$

$$\boxed{\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{6}{7 - a_n}}$$

$$L = \frac{6}{7 - L}$$

$$7L - L^2 = 6$$

$$0 = L^2 - 7L + 6$$

$0 = (L - 6)(L - 1)$

$L = 6 \qquad L = 1$

$a_1 = 2$ dec

Answer is Converge to 1

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

15. Assume the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \frac{3n^2}{n^2 - 10}$.

(a) Compute a_5.

$a_5 = S_5 - S_4$

$= \frac{75}{15} - \frac{3(16)}{6} = 5 - \frac{48}{6}$

$= 5 - 8 = -3$

(b) Determine if $\sum_{n=1}^{\infty} a_n$ converges. If possible, give the sum of the series.

$\text{Sum} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3n^2}{n^2-10}$

$\text{Sum} = 3$

The series converges

16. Determine if the series converges or diverges. Give the sum if convergent.

(a) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

$a_n = \cos\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$

by the test for div the series will dv.

(b) $\sum_{n=5}^{\infty} 10\left(\frac{-2}{5}\right)^{n-1}$

geometric series

$a + ar + ar^2 + ar^3 + ar^4 + \ldots$

$\underbrace{10\left(\frac{-2}{5}\right)^4} + \underbrace{10\left(\frac{-2}{5}\right)^5} + 10\left(\frac{-2}{5}\right)^6 + 10\left(\frac{-2}{5}\right)^7 + \ldots$

$r = \frac{-2}{5}$ if $|r| < 1$ the geometric series will conv.

Conv.

$a = 10\left(\frac{-2}{5}\right)^4$

$= \frac{10 \cdot 16}{5^4} = \frac{160}{5^4}$

$Sum = \frac{a}{1-r} = \frac{\frac{160}{5^4}}{1 - \frac{-2}{5}}$

$= \frac{\frac{160}{5^4}}{\frac{7}{5}} = \frac{160}{5^4} \cdot \frac{5}{7}$

160

$$= \frac{160}{7 \cdot 5^3}$$

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(c) $\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{3n+1}}$ $= \underset{n=1}{\frac{3^3}{2^4}} + \underset{n=2}{\frac{3^4}{2^7}} + \underset{n=3}{\frac{3^5}{2^{10}}} + \underset{n=4}{\frac{3^6}{2^{13}}} + \cdots\cdots$

$r = \frac{3}{2^3} = \frac{3}{8}$

is $|r| < 1$? yes

The series conv.

$a = \frac{3^3}{2^4} = \frac{27}{16}$

$\text{Sum} = \frac{a}{1-r} = \frac{27/16}{1 - 3/8}$

$= \frac{27/16}{5/8} = \frac{27}{16} \cdot \frac{8}{5}$

$= \frac{27}{10}$

(d) $\displaystyle\sum_{k=1}^{\infty}\left[\cos\left(\frac{1}{k+3}\right)-\cos\left(\frac{1}{k+1}\right)\right]$

need S_n

$S_n = \cancel{\cos\left(\frac{1}{4}\right)} - \cos\left(\frac{1}{2}\right)$ K=1

$+ \cancel{\cos\left(\frac{1}{5}\right)} - \cos\left(\frac{1}{3}\right)$ K=2

$+ \cos\left(\frac{1}{6}\right) - \cancel{\cos\left(\frac{1}{4}\right)}$ K=3

$+ \cos\left(\frac{1}{7}\right) - \cancel{\cos\left(\frac{1}{5}\right)}$ K=4

$\vdots$

$\vdots$

$+ \cancel{\cos\left(\frac{1}{n}\right)} - \cos\left(\frac{1}{n-2}\right)$ K=n-3

$+ \cancel{\cos\left(\frac{1}{n+1}\right)} - \cos\left(\frac{1}{n-1}\right)$ K=n-2

$+ \cos\left(\frac{1}{n+2}\right) - \cancel{\cos\left(\frac{1}{n}\right)}$ K=n-1

$+ \cos\left(\frac{1}{n+3}\right) - \cancel{\cos\left(\frac{1}{n+1}\right)}$ K=n

$$S_n = -\cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{3}\right) + \cos\left(\frac{1}{n+2}\right) + \cos\left(\frac{1}{n+3}\right)$$

$$\lim_{n\to\infty} S_n = -\cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{3}\right) + \cos(0) + \cos(0)$$

$$= -\cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{3}\right) + 1 + 1$$

$$= \underbrace{2 - \cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{3}\right)}$$

The series will converge to

$\sum a_n = a_1 + a_2 + a_3 + \cdots\cdots + a_n + \cdots\cdots$

if $\lim_{n \to \infty} a_n = L$ and $L \neq 0$ the series will div.

Test for div.

if $\lim_{n \to \infty} a_n = 0$ may conv or may not conv.

more work

$\sum a_n$ where S_n is a partial sum

if $\lim_{n \to \infty} S_n = a\#$ the the

series converges to that #.