Math 152 Week in Review: Sections 10.2, 10.3

Solutions and questions can be found at the link:
https://www.math.tamu.edu/~kahlig/152WIR.html

1. Find the length of the arc of the curve $x = t^2$, $y = t^3$ that lies between the points $(1, 1)$ and $(4, 8)$.

$t=2$ $\quad$ $t=1$

$4=t^2$ $\quad$ $1=t^2$

$t=\pm 2$ $\quad$ $t=\pm 1$

$1 \le t \le 2$

$$L = \int_a^b ds \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$

$$= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt = \int_1^2 \sqrt{t^2(4+9t^2)}\, dt$$

$$= \int_1^2 |t| \sqrt{4+9t^2}\, dt = \int_1^2 t\sqrt{4+9t^2}\, dt$$

$u = 4+9t^2$

$du = 18t\, dt$

$\frac{1}{18} du = t\, dt$

$t=1$: $u=13$

$t=2$: $u = 4+9(4) = 4+36 = 40$

$$= \int_{13}^{40} \frac{1}{18}\sqrt{u}\, du$$

$3/2 \quad 40 \quad 3/2 \quad 3/2$

$$= \frac{1}{18} \cdot \frac{2}{3} u^{3/2} \Big|_{13}^{40} = \frac{1}{27} (40)^{3/2} - \frac{1}{27} (13)^{3/2}$$

2. Find the length of the curve $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \le t \le 2$ $\longrightarrow$ $4e^{\frac{t}{2}} = 4e^{\frac{1}{2}t}$

$$\frac{dx}{dt} = e^t - 1 \qquad \frac{dy}{dt} = 2e^{\frac{1}{2}t}$$

$$ds = \sqrt{(e^t-1)^2 + \left(2e^{\frac{1}{2}t}\right)^2}\, dt$$

$$= \sqrt{(e^t-1)(e^t-1) + \left(2e^{\frac{1}{2}t}\right)\left(2e^{\frac{1}{2}t}\right)}\, dt$$

$$= \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$= \sqrt{e^{2t} + 2e^t + 1}\, dt$$

$$ds = \sqrt{(e^t+1)^2}\, dt = (e^t+1)\,dt$$

$$L = \int_0^2 ds = \int_0^2 (e^t+1)\, dt$$

$$= (e^t + t)\Big|_0^2$$

$$= e^2 + 2 - (e^0 + 0)$$

$$= e^2 + 2 - 1 = \underline{e^2 + 1}$$

3. Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ for $1 \le x \le 3$

$\frac{1}{2t} = \frac{1}{2} t^{-1}$

$x = t \qquad y = \frac{t^3}{6} + \frac{1}{2t} \qquad 1 \le t \le 3$

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = \frac{3t^2}{6} - \frac{1}{2}t^{-2} = \frac{t^2}{2} - \frac{1}{2t^2}$

$$ds = \sqrt{(1)^2 + \left(\frac{t^2}{2} - \frac{1}{2t^2}\right)^2}\, dt$$

$$= \sqrt{1 + \left(\frac{t^2}{2} - \frac{1}{2t^2}\right)\left(\frac{t^2}{2} - \frac{1}{2t^2}\right)}\, dt$$

$$= \sqrt{1 + \frac{t^4}{4} - 2\left(\frac{1}{4}\right) + \frac{1}{4t^4}}\, dt$$

$$= \sqrt{1 + \frac{t^4}{4} - \frac{1}{2} + \frac{1}{4t^4}}\, dt$$

$$= \sqrt{\frac{t^4}{4} + \frac{1}{2} + \frac{1}{4t^4}}\, dt$$

$$ds = \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t^2}\right)^2}\, dt = \left(\frac{t^2}{2} + \frac{1}{2t^2}\right) dt$$

$1 \quad 1 \qquad 1 \quad , -2$

$$\frac{1}{2t^2} = \frac{1}{2} t^{-2}$$

$$L = \int_1^3 ds = \int_1^3 \frac{t^2}{2} + \frac{1}{2t^2}\, dt$$

$$= \left(\frac{t^3}{6} + \frac{1}{2} \frac{t^{-1}}{-1} \right) \Bigg|_1^3 = \left(\frac{t^3}{6} - \frac{1}{2t} \right) \Bigg|_1^3$$

$$= \frac{27}{6} - \frac{1}{6} - \left(\frac{1}{6} - \frac{1}{2} \right)$$

$$= \frac{27}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{2} = \underbrace{\frac{25}{6} + \frac{1}{2}}$$

$$L = \int_a^b ds$$

$$SA = \int_a^b 2\pi r \, ds$$

about x-axis

$$SA = \int_a^b 2\pi y \, ds$$

about the y-axis

$$SA = \int_a^b 2\pi x \, ds$$

4. Find the area of the surface obtained by rotating the curve about the y−axis.

$x = 5\sin t, \quad y = 5\cos t, \quad 0 \le t \le \pi$

$$\frac{dx}{dt} = 5\cos t \qquad \frac{dy}{dt} = -5\sin(t)$$

$$SA = \int_0^{\pi} 2\pi x \, ds = \int_0^{\pi} 2\pi \, 5\sin(t) \sqrt{(5\cos t)^2 + (-5\sin t)^2} \, dt$$

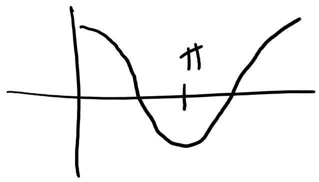
$$= \int_0^{\pi} 10\pi \sin(t) \sqrt{25\cos^2 t + 25\sin^2 t} \, dt$$

$$= \int_0^{\pi} 10\pi \sin(t) \sqrt{25(\cos^2 t + \sin^2 t)} \, dt$$

$$= \int_0^{\pi} 10\pi \sin(t) \sqrt{25} \, dt = \int_0^{\pi} 50\pi \sin(t) \, dt$$

$$= 50\pi \left(-\cos(t)\right)\Big|_0^{\pi} = 50\pi\left[-\cos(\pi) - -\cos(0)\right]$$

$$= 50\pi \left(-\cos(t)\right)\Big|_0^{\pi} = 50\pi\left[-\cos(\pi) - -\cos(0)\right]$$



$$= 50\pi\left[-(-1) + (1)\right] = 50\pi(2)$$

$$= 100\pi$$

5. Find the area of the surface obtained by rotating the curve about the x-axis.

$x = \frac{t^3}{3}$, $y = t^2$, $0 \le t \le 1$

$\frac{dx}{dt} = \frac{3t^2}{3} = t^2$ $\frac{dy}{dt} = 2t$

$SA = \int_0^1 2\pi y \, ds = \int_0^1 2\pi (t^2) \sqrt{(t^2)^2 + (2t)^2} \, dt$

$= \int_0^1 2\pi t^2 \sqrt{t^4 + 4t^2} \, dt$

$= \int_0^1 2\pi t^2 \sqrt{t^2(t^2+4)} \, dt$

$= \int_0^1 2\pi t^2 \cdot t \sqrt{t^2+4} \, dt$

$u = t^2+4$

$du = 2t\,dt$

$\frac{1}{2} du = t\,dt$

$= \int_4^5 2\pi (u-4) \cdot \frac{1}{2} \sqrt{u} \, du$

$t^2 = u-4$

if $t=0$
$u=4$

$= \pi \int^5 \left(u^{3/2} - 4u^{1/2}\right) du$

if $t=1$
$u=5$

$$= \pi \int_4 \left(u^{3/2} - 4u^{1/2} \right) du \qquad \text{if } t = 1, \; u = 5$$

$$= \pi \left[\frac{2}{5} u^{5/2} - 4 \cdot \frac{2}{3} u^{3/2} \right] \Bigg|_4^5$$

$$= \pi \left(\frac{2}{5}(5)^{5/2} - \frac{8}{3}(5)^{3/2} - \left[\frac{2}{5}(4)^{5/2} - \frac{8}{3}(4)^{3/2} \right] \right)$$

6. Setup the integral that would find the area of the surface obtained by rotating the curve given by the parametric equations given below on the interval $0 \le t \le 2$.

$x = 2t - t^2$ $\quad$ $y = 3 + t^2$ $\quad$ $\frac{dx}{dt} = (2 - 2t)$ $\quad$ $\frac{dy}{dt} = 2t$

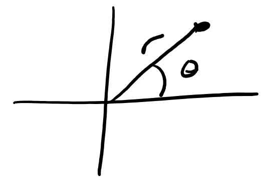
(a) about the x-axis.

$$\int_0^2 2\pi y \, ds = \int_0^2 2\pi(3+t^2)\sqrt{(2-2t)^2 + (2t)^2} \, dt$$

(b) about the y-axis.

$$\int_0^2 2\pi x \, ds = \int_0^2 2\pi(2t-t^2)\sqrt{(2-2t)^2 + (2t)^2} \, dt$$

(x, y) (r, θ)

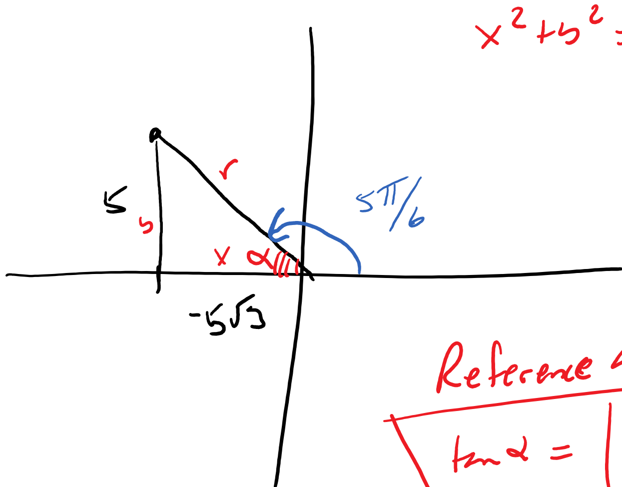


7. Find two other pairs of polar coordinates to the given polar coordinate, one with $r > 0$ and one with $r < 0$ and both with $0 \le \theta < 2\pi$

$x^2 + y^2 = r^2$

Cartesian point

$(-5\sqrt{3}, 5)$



$(-5\sqrt{3})^2 + (5)^2 = r^2$

$25(3) + 25 = r^2$

$75 + 25 = r^2$

$r^2 = 100$

$r = 10$

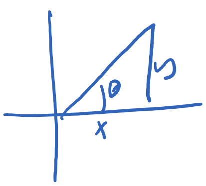
Reference angle

$\tan\alpha = \left|\frac{y}{x}\right|$

$\tan\alpha = \frac{5}{5\sqrt{3}}$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = 30° = \frac{\pi}{6}$



$\tan\theta = \frac{y}{x}$

$\theta = \arctan\left(\frac{y}{x}\right)$

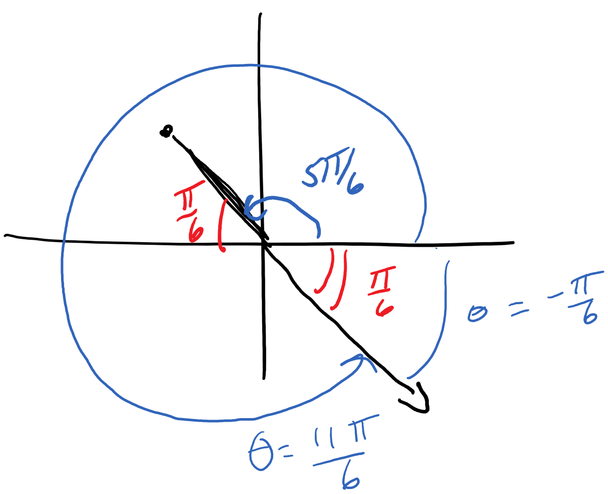
$r > 0$

$\left(10, \frac{5\pi}{6}\right)$

$r < 0$

doesn't match the interval

$\left(-10, \frac{-\pi}{6}\right)$



$r < 0$

$\left(-10, \frac{11\pi}{6}\right)$

8. Find the Cartesian coordinates of the polar point $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$

$r \quad \theta$

$x = r\cos\theta$

$y = r\sin\theta$

$x = 4\sqrt{2}\cos\left(\frac{3\pi}{4}\right)$

$x = 4\sqrt{2} \cdot \left(-\frac{\sqrt{2}}{2}\right)$

$x = 4\left(-\frac{2}{2}\right)$

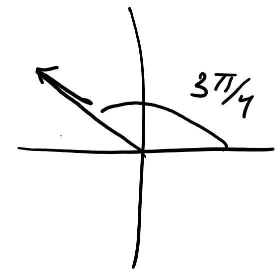
$x = -4$

$y = 4\sqrt{2}\sin\left(\frac{3\pi}{4}\right)$

$y = 4\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)$

$y = 4 \cdot \left(\frac{2}{2}\right)$

$y = 4$



$\boxed{\text{point } (-4, 4)}$

~~9.~~ Give two polar representations for the point $(-5\sqrt{3}, 5)$. One with $r > 0$ and one with $r < 0$.

This was Really Question 7.

10. Write a Cartesian equation for the polar curve $r = -8\sin\theta$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$

$r^2 = -8r\sin\theta$

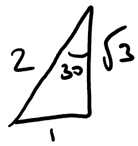
$\boxed{x^2+y^2 = -8y}$

11. Write a Cartesian equation for the polar curve $r^2\sin(2\theta) = 1$.

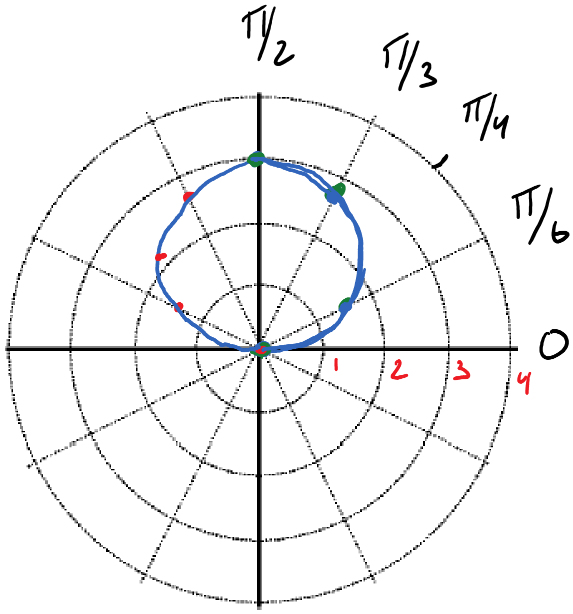
$r^2 \cdot 2\sin\theta\cos\theta = 1$

$2r\sin\theta\, r\cos\theta = 1$

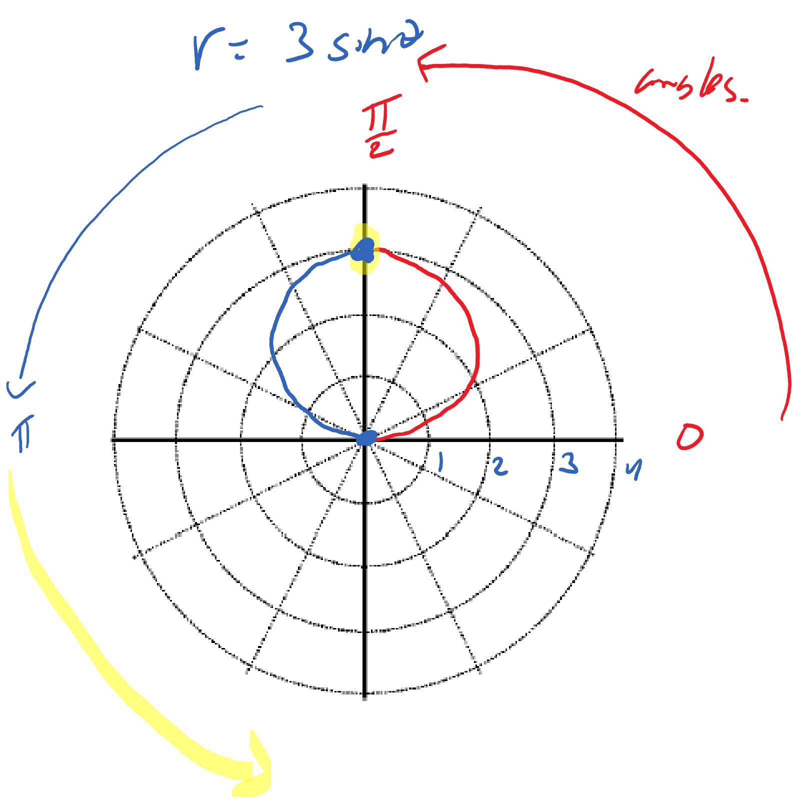
$2yx = 1 \longrightarrow y = \frac{1}{2x}$

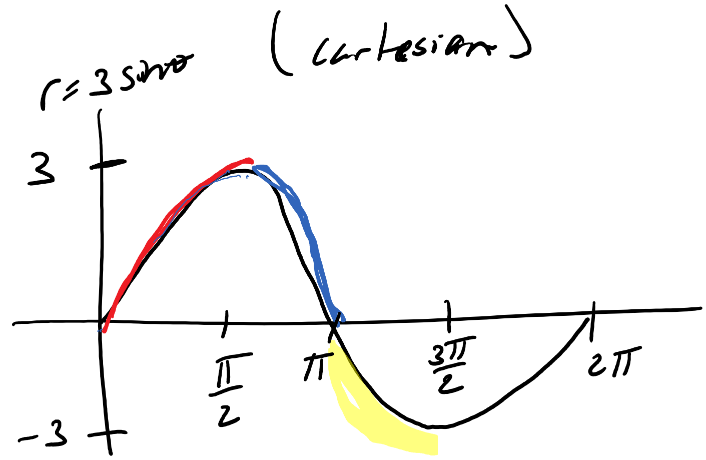


12. Graph $r = 3\sin(\theta)$



θ	r
0	0
$\frac{\pi}{6}$	$3\left(\frac{1}{2}\right) = \frac{3}{2} = 1.5$
$\frac{\pi}{4}$	$3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2} = 2.1213$
$\frac{\pi}{3}$	$3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2} = 2.598$
$\frac{\pi}{2}$	$3(1) = 3 = 3$
$\frac{2\pi}{3}$	2.598
$\frac{3\pi}{4}$	2.1213
$\frac{5\pi}{6}$	1.5
π	0

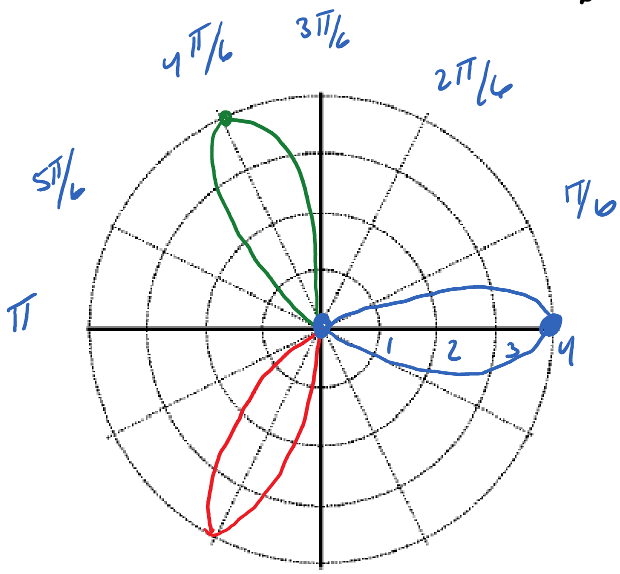


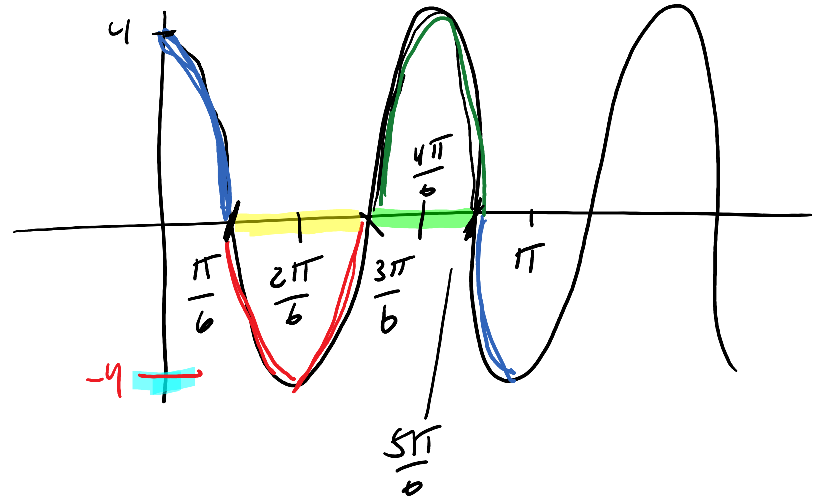


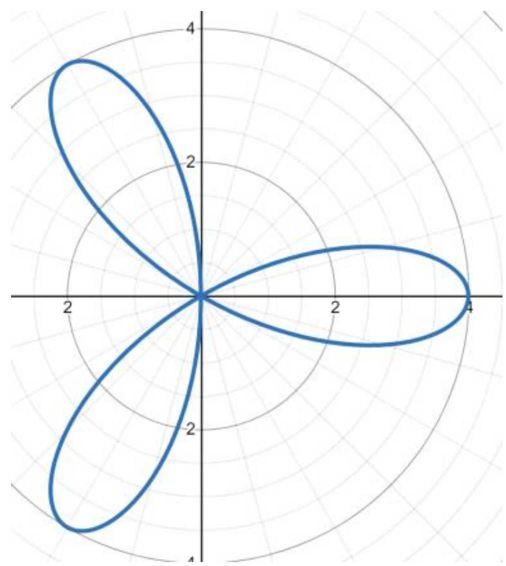
$3\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{6}$

13. Graph $r = 4\cos(3\theta)$





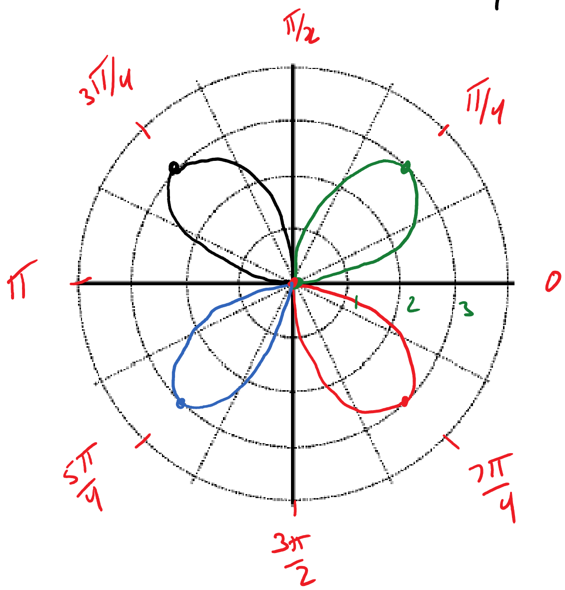


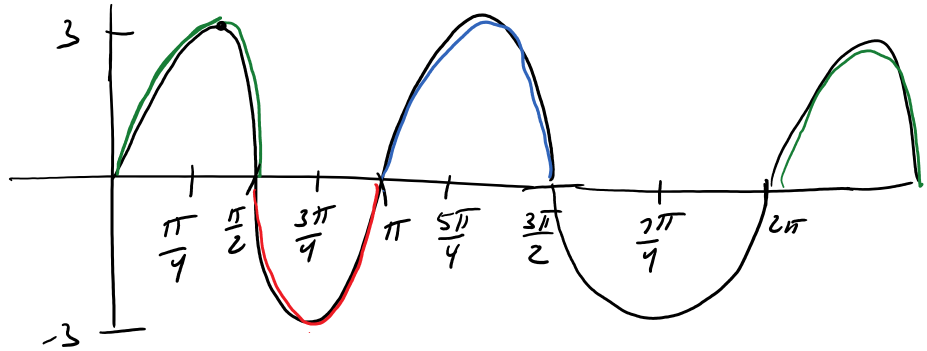
Screen clipping taken: 4/26/2021 8:26 PM

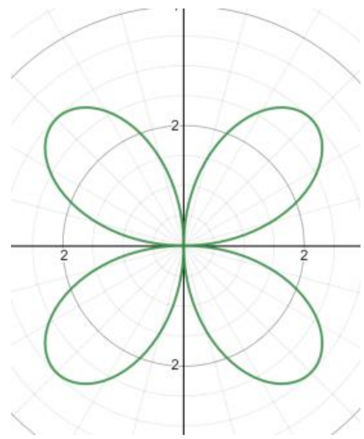
14. Graph $r = 3\sin(2\theta)$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$



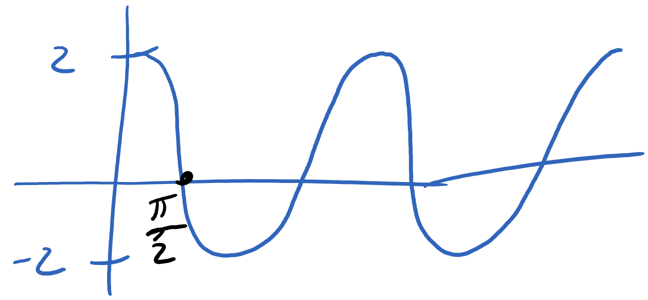




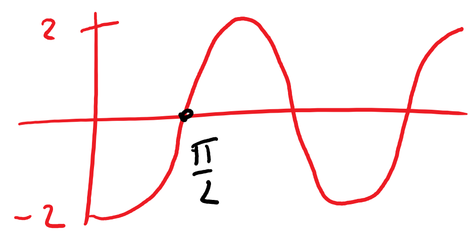
Screen clipping taken: 4/26/2021 8:33 PM

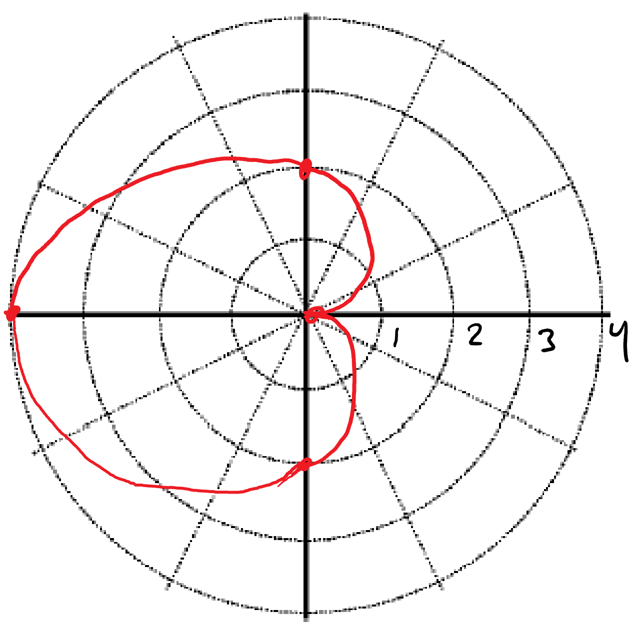
15. Graph $r = 2 - 2\cos\theta$

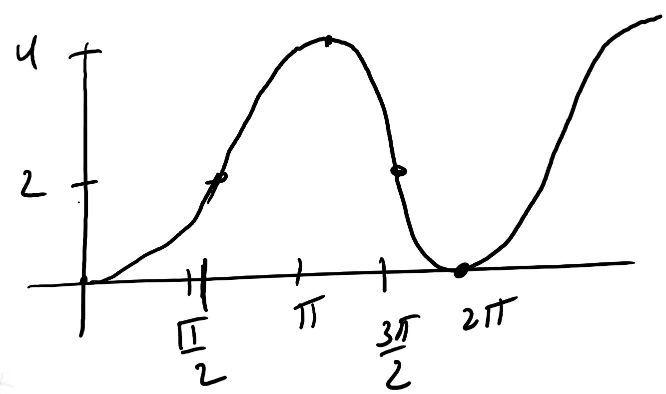
$2\cos\theta$

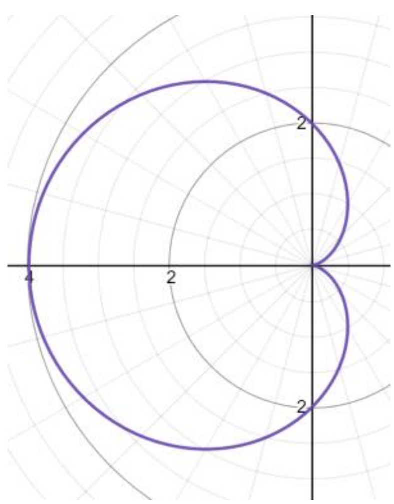


$-2\cos\theta$





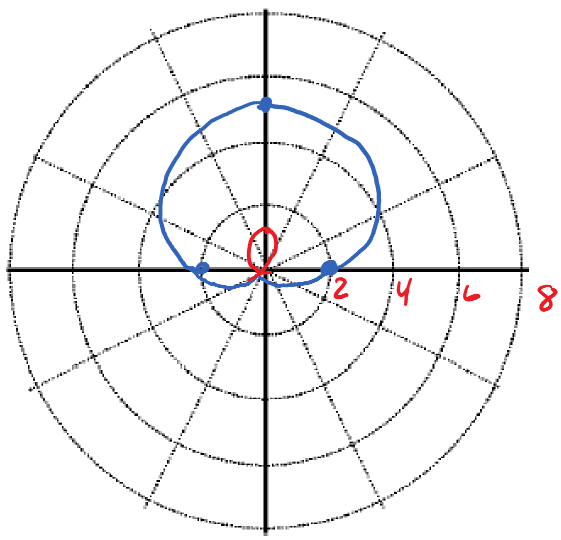


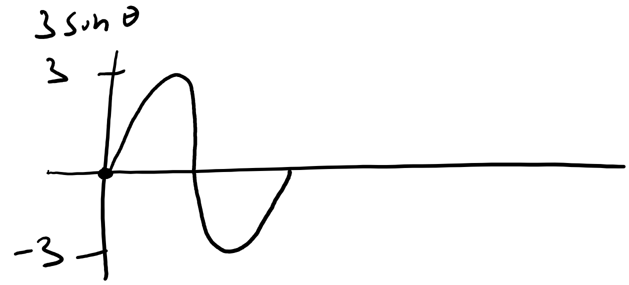


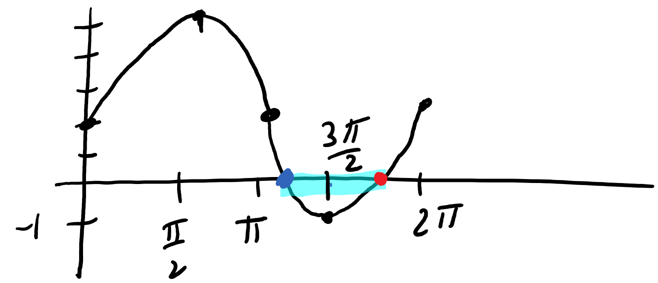
Screen clipping taken: 4/26/2021 8:41 PM

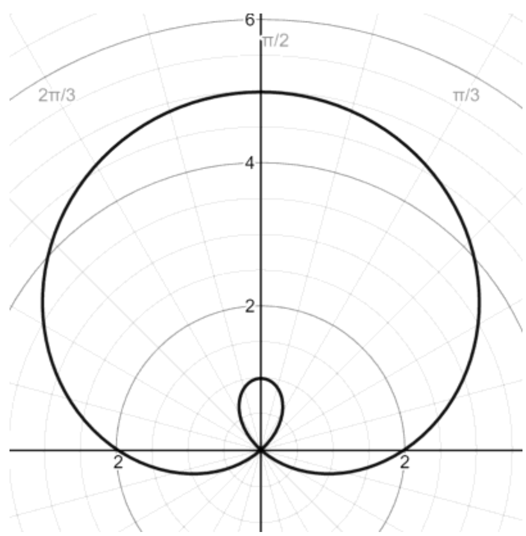
16. Graph $r = 2 + 3\sin\theta$

$r \to 0$









Screen clipping taken: 4/26/2021 8:46 PM