152 Week In Review: Section 7.3

Solutions and questions can be found at the link:
https://www.math.tamu.edu/~kahlig/152WIR.html

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

$$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$\int \csc^3 x \, dx = \frac{-1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

1. $\int \frac{x^3}{\sqrt{9-x^2}}\,dx =$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

need $x^2 = 9\sin^2\theta$

$9\cos^2\theta = 9 - 9\sin^2\theta$

Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^3}{\sqrt{9-9\sin^2\theta}}\, 3\cos\theta\, d\theta$$

$$= \int \frac{(3\sin\theta)^3}{\sqrt{9\cos^2\theta}} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{3^3\sin^3\theta \cdot \cancel{3\cos\theta}}{\cancel{3\cos\theta}}\, d\theta$$

$$= 27\int \sin^3\theta\, d\theta$$

$$= 27\int \sin^2\theta\, \sin\theta\, d\theta = 27\int (1-\cos^2\theta)\sin\theta\, d\theta$$

$u = \cos\theta$ $\quad du = -\sin\theta\, d\theta$

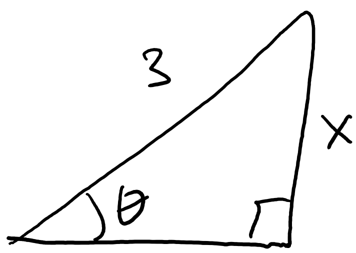
$-du = \sin\theta\, d\theta$

$-du = \sin\theta$

$$= -27\int 1-u^2\,du = -27\left[u - \frac{u^3}{3}\right] + C$$

$x = 3\sin\theta$

$\sin\theta = \dfrac{x}{3}$



$\sqrt{3^2 - x^2}$

$= \sqrt{9 - x^2}$

$$= -27\left[\cos\theta - \frac{1}{3}\cos^3(\theta)\right] + C$$

$$= -27\left[\frac{\sqrt{9-x^2}}{3} - \frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3\right] + C$$

$$= -9\sqrt{9-x^2} + 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C$$

2. $\int \frac{\sqrt{4x^2 - 25}}{x^4}\, dx =$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$25\tan^2\theta = 25\sec^2\theta - 25$

need $4x^2 = 25\sec^2\theta$

Let $2x = 5\sec\theta$

$x = \frac{5}{2}\sec\theta$

$dx = \frac{5}{2}\sec\theta\tan\theta\, d\theta$

$$\int \frac{\sqrt{4x^2-25}}{x^4}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{\left(\frac{5}{2}\sec\theta\right)^4} \cdot \frac{5}{2}\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\sqrt{25\tan^2\theta}}{\left(\frac{5}{2}\right)^4 \sec^4\theta} \cdot \frac{5}{2}\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{5\tan\theta}{\left(\frac{5}{2}\right)^3 \sec^3\theta} \tan\theta\, d\theta$$

$$= \int \frac{5\tan^2\theta}{\frac{125}{8}\sec^3\theta}\, d\theta = \frac{5\cdot 8}{125}\int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$$

$$= \frac{8}{25} \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^3\theta \, d\theta$$

$$= \frac{8}{25} \int \sin^2\theta \cos\theta \, d\theta$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= \frac{8}{25} \int u^2 \, du$$

$$= \frac{8}{25} \cdot \frac{1}{3} u^3 + C$$

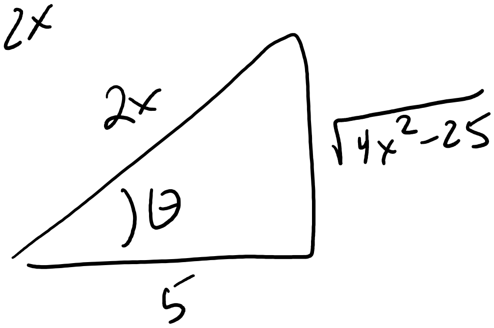
$$= \frac{8}{75} \sin^3\theta + C$$

$$= \frac{8}{75} \left(\frac{\sqrt{4x^2-25}}{2x} \right)^3 + C$$

$2x = 5\sec\theta$

$\frac{2x}{5} = \sec\theta$

$\frac{5}{2x} = \cos\theta$



3. $\displaystyle\int \frac{x^3}{\sqrt{x^2+25}}\,dx =$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

need $x^2 = 25\tan^2\theta$

$25\sec^2\theta = 25 + 25\tan^2\theta$

Let $x = 5\tan\theta$

$dx = 5\sec^2\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{x^2+25}}\,dx = \int \frac{(5\tan\theta)^3}{\sqrt{25\tan^2\theta+25}} \cdot 5\sec^2\theta\, d\theta$$

$$= \int \frac{5^3\tan^3\theta \cdot 5\sec^2\theta}{\sqrt{25\sec^2\theta}}\, d\theta$$

$$= \int \frac{5^3\tan^3\theta\; 5\sec^2\theta}{5\sec\theta}\, d\theta$$

$$= \int 5^3\tan^3\theta\,\sec\theta\, d\theta$$

$$= 5^3 \int \tan^2\theta\;\tan\theta\sec\theta\, d\theta$$

$$= 5^3 \int (\sec^2\theta - 1)\,\tan\theta\sec\theta\, d\theta$$

$u = \sec\theta$

$$= 5^3 \int u^2 - 1\, du$$

$u = \sec\theta$

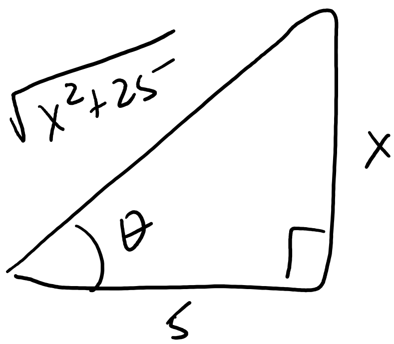
$du = \sec\theta \tan\theta \, d\theta$

$= 5^3 \int u^2 - 1 \, du$

$= 5^3 \left[\frac{1}{3} u^3 - u \right] + C$

$x = 5\tan\theta$

$\frac{x}{5} = \tan\theta$



$= 5^3 \left[\frac{1}{3} \sec^3\theta - \sec\theta \right] + C$

$= 5^3 \left[\frac{1}{3} \cdot \left(\frac{\sqrt{x^2+25}}{5} \right)^3 - \left(\frac{\sqrt{x^2+25}}{5} \right) \right] + C$

4. $\int \frac{1}{(1+x^2)^2}\,dx =$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$\sec^2\theta = 1 + \tan^2\theta$

need $x^2 = \tan^2\theta$

let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\, d\theta$$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\, d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$x = \tan\theta$

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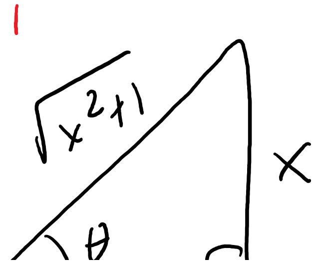
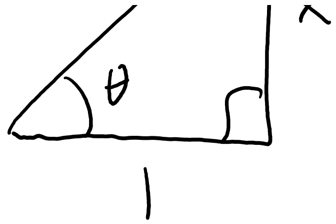


$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$



$$= \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\arctan(x) + \frac{x}{\sqrt{x^2+1}}\,\frac{1}{\sqrt{x^2+1}}\right] + C$$

$\theta = \arctan(x)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{1}{2}\left(\arctan(x) + \frac{x}{x^2+1}\right] + C$$

5. $\displaystyle\int \frac{x^2}{(4-9x^2)^{3/2}}\,dx =$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

need $9x^2 = 4\sin^2\theta$

$4\cos^2\theta = 4 - 4\sin^2\theta$

Let $3x = 2\sin\theta$

$x = \frac{2}{3}\sin\theta$

$dx = \frac{2}{3}\cos\theta\,d\theta$

$$\int \frac{\left(\left(\frac{2}{3}\right)\sin\theta\right)^2}{\left(4-4\sin^2\theta\right)^{3/2}} \cdot \frac{2}{3}\cos\theta\,d\theta$$

$$= \int \frac{\left(\frac{2}{3}\right)^2 \sin^2\theta \cdot \frac{2}{3}\cos\theta}{\left(4\cos^2\theta\right)^{3/2}}\,d\theta$$

$$= \int \frac{\left(\frac{2}{3}\right)^3 \sin^2\theta\cos\theta}{2^3\cos^3\theta}\,d\theta$$

$$= \int \frac{2^3}{3^3} \cdot \frac{\sin^2\theta}{2^3\cos^2\theta}\,d\theta = \int \frac{1}{27}\,\frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

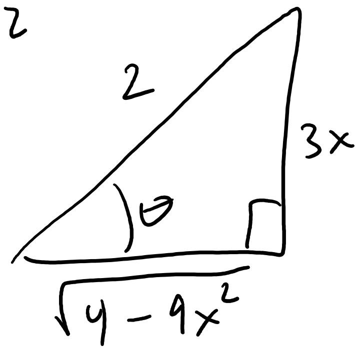
$$= \frac{1}{} \int \tan^2\theta\,d\theta = \frac{1}{27}\int \sec^2\theta - 1\,d\theta$$

$$= \frac{1}{27}\int \tan^2\theta \, d\theta = \frac{1}{27}\int \sec^2\theta - 1 \, d\theta$$

$$= \frac{1}{27}\left[\tan\theta - \theta\right] + C$$

$3x = 2\sin\theta$

$\frac{3x}{2} = \sin\theta$



$$= \frac{1}{27}\left(\frac{3x}{\sqrt{4-9x^2}} - \arcsin\left(\frac{3x}{2}\right)\right) + C$$

6. Complete the square for $x^2 + 8x + 3$ $= \underbrace{x^2 + 8x + \underline{4}^2}\ - \underline{4}^2 + 3$

$\frac{1}{2}(8) = 4$

$= (x+4)^2 - 16 + 3$

$= (x+4)^2 - 13$

7. complete the square for $5x^2 - 40x + 4$ $= 5\left[x^2 - 8x\right] + 4$

factor.

$\frac{1}{2}(8) = 4$

$= 5\left[x^2 - 8x + \underline{4}^2 - \underline{4}^2\right] + 4$

$= 5\left[(x-4)^2 - 16\right] + 4$

$= 5(x-4)^2 - 80 + 4$

$= 5(x-4)^2 - 76$

8. $\int \frac{1}{\sqrt{x^2+2x+10}}\,dx$

Step 1 complete the square

$$x^2+2x+10 = \left[x^2+2x+1^2-1^2\right]+10$$

$\frac{1}{2}(2)=1$

$$= (x+1)^2 - 1 + 10 = (x+1)^2+9$$

$$\int \frac{1}{\sqrt{x^2+2x+10}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+9}}\,dx$$

need $(x+1)^2 = 9\tan^2\theta$

$\cos^2\theta = 1-\sin^2\theta$

$\sec^2\theta = 1+\tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$9\sec^2\theta = 9 + 9\tan^2\theta$

let $x+1 = 3\tan\theta$

$x = -1+3\tan\theta$

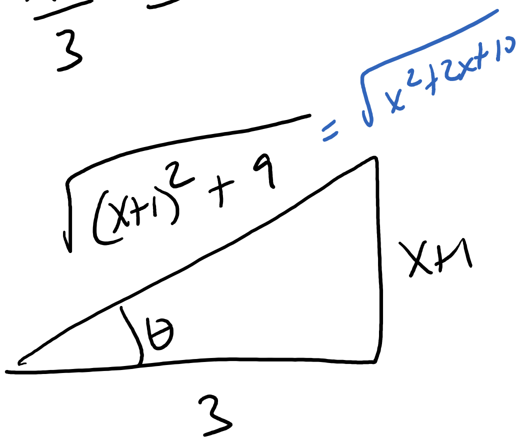
$dx = 3\sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{9\tan^2\theta+9}}\cdot 3\sec^2\theta\,d\theta = \int \frac{3\sec^2\theta}{\sqrt{9\sec^2\theta}}\,d\theta$$

$$= \int \frac{3\sec^2\theta}{3\sec\theta}\,d\theta = \int \sec\theta\,d\theta$$

$x+1 = 3\tan\theta$

$\frac{x+1}{3} = \tan\theta$



$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+2x+10}}{3} + \frac{x+1}{3}\right| + C$$

9. $\int \frac{1}{(x+3)^2\sqrt{x^2+6x+5}}\,dx =$

$$x^2+6x+5 = \left[x^2+6x+\underline{3^2}-\underline{3^2}\right]+5$$

$\frac{1}{2}(6) = 3$

$$= (x+3)^2 - 9 + 5 = (x+3)^2 - 4$$

$$\int \frac{1}{(x+3)^2\sqrt{(x+3)^2-4}}\,dx$$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$4\tan^2\theta = 4\sec^2\theta - 4$

need $(x+3)^2 = 4\sec^2\theta$

let $x+3 = 2\sec\theta$

$x = -3 + 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

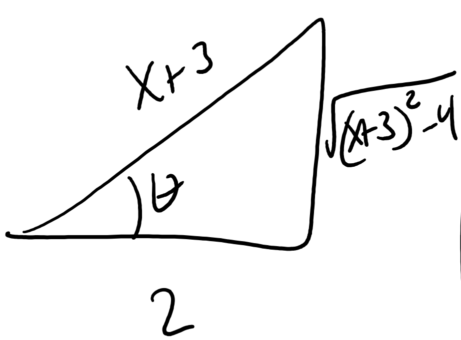
$$\int \frac{1}{4\sec^2\theta\sqrt{4\sec^2\theta-4}} \cdot 2\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{2\sec\theta \tan\theta}{4\sec^2\theta \sqrt{4\tan^2\theta}}\, d\theta = \int \frac{\tan\theta}{2\sec\theta \cdot 2\tan\theta}\, d\theta$$

$$= \int \frac{1}{4\sec\theta}\, d\theta = \int \frac{1}{4} \cos\theta \, d\theta$$

$x+3 = 2\sec\theta$

$\frac{x+3}{2} = \sec\theta$



$$= \frac{1}{4} \sin\theta + C$$

$$= \frac{1}{4} \frac{\sqrt{(x+3)^2-4}}{x+3} + C$$