

## Section 1.2: Accumulation/Amount Functions

### Chapter 1: The Measurement of Interest

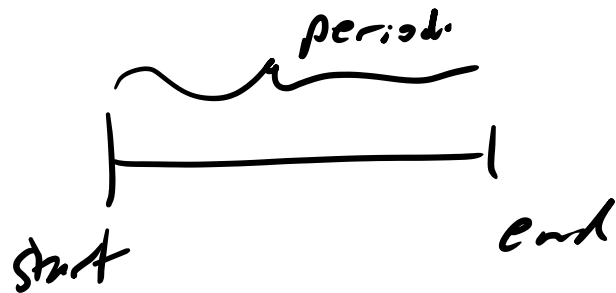
**Interest** may be defined as the compensation that a borrower of capital pays to a lender of capital for its use.

**Principal** is the initial amount of money invested.

**Accumulated amount** The total amount of money received after a period of time.

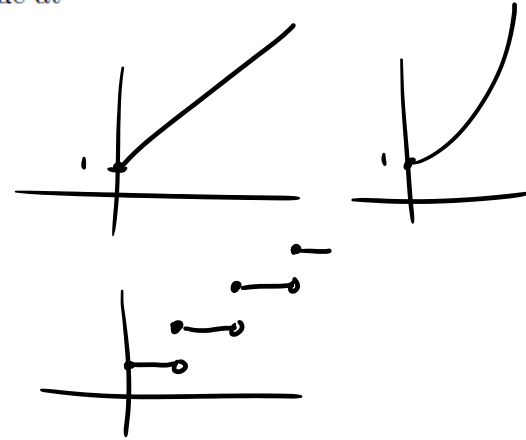
The amount of **interest** earned is the difference between the accumulated amount and the principal.

The **period** is the unit in which time is measured. The more common measurement period is one year, and will be assumed unless otherwise stated.



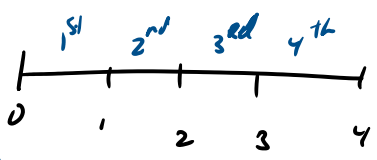
The accumulation function, denoted  $a(t)$ , gives the accumulated value at time  $t \geq 0$  of an original investment of 1. The properties of  $a(t)$  are:

- $a(0) = 1$
- For positive interest rates,  $a(t)$  in an increasing function. Negative interest rates gives a decreasing function.
- If interest accrues continuously, the function will be continuous. Otherwise,  $a(t)$  will have discontinuities.



The amount function, denoted  $A(t)$ , gives the accumulated value at time  $t \geq 0$  of an original investment of  $k$ . The properties of  $A(t)$  are:

- $A(t) = k * a(t)$  so we get  $A(0) = k$
- For positive interest rates,  $A(t)$  in an increasing function. Negative interest rates gives a decreasing function.
- If interest accrues continuously, the function will be continuous. Otherwise,  $A(t)$  will have discontinuities.



Define  $I_n$  as the amount of interest earned in the  $n$ -th period from the date of investment.

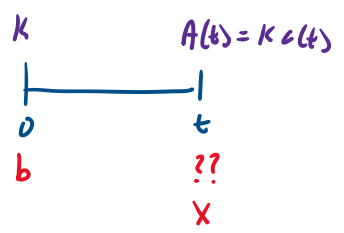
$I_n = A(n) - A(n - 1)$  for  $n = 1, 2, 3, \dots$

$I_1 = A(1) - A(0)$   
 $I_2 = A(2) - A(1)$

Proportionality: Suppose an investment of  $b$  is made and this investment will follow another investment strategy whose amount function is given by  $A(t)$ ,  $t > 0$ . Assume no other deposits or withdrawals for this investment.

$A(t) = K a(t)$

- If the investment of  $b$  is made at time 0, then the value of the investment at time  $t$  is given by  $\frac{b \cdot A(t)}{A(0)}$ .

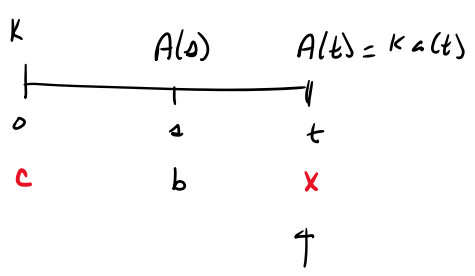


$X = \frac{b \cdot A(t)}{A(0)}$

$X = b a(t) = \frac{b \cdot K a(t)}{K} = \frac{b \cdot K a(t)}{K a(0)}$

- If the investment of  $b$  is made at time  $s$ , then the value of the investment at time  $t$  is given by  $\frac{b \cdot A(t)}{A(s)}$ .

$A(t) = K a(t)$



$b = c a(s)$   $\rightarrow c = \frac{b}{a(s)}$

$X = c a(t) = \frac{b}{a(s)} a(t)$

$= \frac{b \cdot K a(t)}{K a(s)} = \frac{b A(t)}{A(s)}$

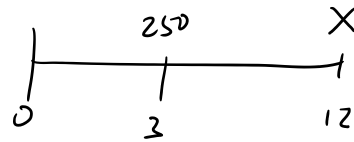
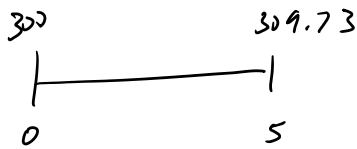
Example: An investment of \$10,000 is made into a fund at time  $t = 0$ . The fund develops the following balances over the next 4 years.

$t$	$A(t)$	$I_n$
0	10,000	
1	10,600	$I_1 = 600$
2	11,024	$I_2 = 424 = 11024 - 10600$
3	11,575.2	$I_3 = 551.2$
4	12,732.72	$I_4 = 1,157.52$

If \$5,000 is invested at  $t = 1$  under the same interest environment, find the accumulated value of the \$5,000 at time  $t = 3$ .

$$\begin{aligned} \text{Balance at } t=3 \text{ is } & \frac{5000 A(3)}{A(1)} = \frac{5000 \cdot 11575.2}{10600} \\ & = \$5460 \end{aligned}$$

Example: It is known that  $a(t)$  is of the form  $a(t) = be^{0.1t} + c$ . If \$300 invested at time 0 accumulates to \$309.73 at time 5, find the accumulated value at time 12 of \$250 invested at time 3.



$$a(0) = 1 \quad A(t) = 300 a(t)$$

$$A(0) = 300$$

$$A(5) = 309.73$$

$$a(0) = 1 = be^0 + c$$

$$1 = b + c$$

$$c = 1 - b$$

$$309.73 = 300 [be^{-1(5)} + c]$$

$$309.73 = 300 [be^{-.5} + 1 - b]$$

$$309.73 = 300be^{-.5} + 300 - 300b$$

$$9.73 = b [300e^{-.5} - 300]$$

$$b = \frac{9.73}{300e^{-.5} - 300} = .044995 \approx .05$$

$$c = 1 - b = .95$$

$$X = \frac{250 A(12)}{A(3)}$$

$$a(t) = .05e^{.1t} + .95$$

$$A(t) = 300 a(t)$$

$$X = \frac{250 A(12)}{A(3)}$$

$$= \underline{\underline{274.20}}$$

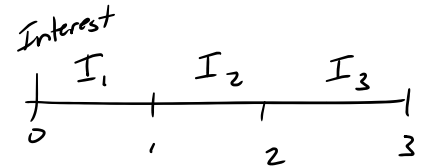
Section 1.3: The Effective Rate of Interest

The effective rate of interest,  $i$ , is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period. This assumes the principal remains constant throughout the period.

Note: The effective rate of interest is often expressed as a percentage. Thus  $7\% \equiv 0.07$  earned per unit of principal.

$$a(0) = 1$$

$$a(1) = 1+i$$



The effective rate of interest for a period can be defined in terms of the amount function by the following.

$$i = \frac{a(1) - a(0)}{a(0)} = \frac{K a(1) - K a(0)}{K a(0)} = \frac{A(1) - A(0)}{A(0)} = \frac{I_1}{A(0)}$$

The effective rate of interest can be computed during the  $n$ -th period.

$$i_n = \frac{I_n}{A(n-1)} \quad \text{for } n=1, 2, 3, \dots$$

The eff. Rate of interest for a period is Ratio of the Interest earned in the period to the Balance at the beginning of the period.

Example: An investment of \$10,000 is made into a fund at time  $t = 0$ . The fund develops the following balances over the next 4 years. Find the effective rate of interest for each of the 3 years.

$t$	$A(t)$	$I_n$	$i_n$
0	10,000		
$\rightarrow$ 1	10,600	$I_1 = 600$	$i_1 = \frac{I_1}{A(0)} = \frac{600}{10000} = .06 \rightarrow 6\%$
2	11,024	$I_2 = 424$	$i_2 = \frac{I_2}{A(1)} = \frac{424}{10600} \rightarrow 4\%$
?? .. 3	11,575.2	$I_3 = 551.2$	$i_3 = \frac{I_3}{A(2)} = \frac{551.2}{11024} \rightarrow 5\%$

$$5000 + 5000(1.04) = \text{Balance at } t=2 \quad 5000(1 + .04)$$

$$\text{at } t=3 \quad 5000(1.04)(1.05) = 5460$$

Section 1.4: simple interest

Simple interest is when the amount of interest earned during each period is constant.

account value	1	1+i	1+2i	1+3i	1+4i
Period	0	1	2	3	4

$$P(1+it) = F = A$$

Thus  $a(t) = 1 + it$  for  $t = 1, 2, 3, \dots$

$$A(t) = K a(t) = K(1+it)$$

Note: A constant rate of simple interest does not imply a constant effective rate of interest.

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{1+in - [1+i(n-1)]}{1+i(n-1)} = \frac{1+in - [1+in-i]}{1+i(n-1)} = \frac{i}{1+i(n-1)}$$

Note: Unless stated otherwise, under simple interest it will be assumed that interest is accrued proportionally over fraction periods.

$$\begin{aligned} \text{As } n \rightarrow \infty \\ i_n \rightarrow 0 \end{aligned}$$



Example: A bank pays a simple interest rate of 2.5% per annum. \$2,000 is deposited on January 1, 2004.

- (a) Compute the accumulated value on April 1, 2006.  
 (b) How long until the accumulated amount is \$2,230?

A)  $1/1/04 \rightarrow 4/1/06 \Rightarrow 2 \text{ yrs } 3 \text{ months.} \rightarrow 2 + \frac{3}{12} = 2 + \frac{1}{4} = \frac{9}{4} \text{ yrs.}$   
 $= 2.25 \text{ yrs.}$

$$\begin{aligned} \text{Balance} &= 2000 (1 + (0.025)(2.25)) \\ &= 2112.50 \end{aligned}$$

B)

$$2300 = 2000 [1 + .025 t]$$

:

$$t = 4.6 \text{ yrs.}$$

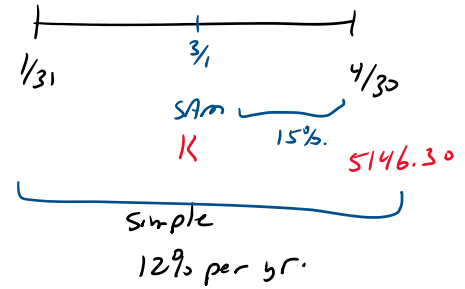
use the yr. 2019.

Example: On January 31 Bob borrows \$5,000 from David and gives David a promissory note. The note states that the loan will be repaid on April 30 of the same year, with interest of 12% per annum. On March 1, David sells the promissory note to Sam, who pays David a sum of money in return for the right to collect the payment from Bob on April 30. Sam pays David an amount such that Sam's yield (interest rate earned) from March 1 to the maturity date can be stated as an annual rate of interest of 15%.

- (a) Determine the amount that Bob was to have paid David on April 30.  
 (b) Determine the amount that Sam paid to David and the simple interest rate David earned on an annual basis.

Bob

5000



A)  $i = .12$  find  $t$ .

$dbd(1.3119, 4.3019) = 89$   
m.m. dd by

$$t = \frac{89}{365}$$

$$5000 \left[ 1 + \frac{89}{365} (.12) \right] = 5146.30$$

Actual methodBob pays back.

B)  $dbd(3.0119, 4.3019) = 60$        $t = \frac{60}{365}$

$$K \left[ 1 + (.15) \frac{60}{365} \right] = 5146.30$$

$$K = 5022.46$$

from David's view

5000

5022.46

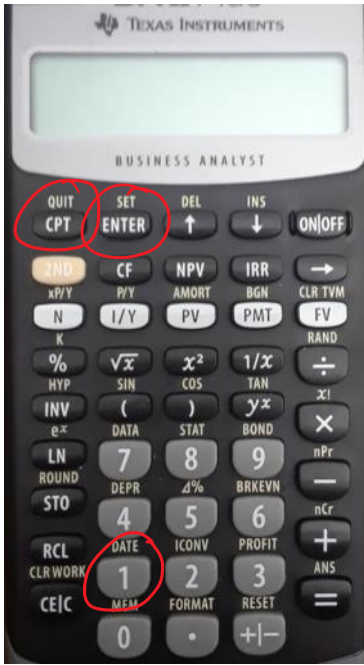


24 days.

$$5000 \left[ 1 + i \frac{29}{365} \right] = 5022.46$$

⋮

$i = 5.65\%$   
 annual simple  
 interest rate.



↑  
actual # of days.



↑  
# of days based on a 360 day yr.

press 2<sup>nd</sup> enter. to switch for the screens.

① 2<sup>nd</sup> 1

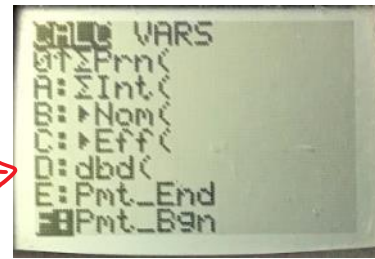
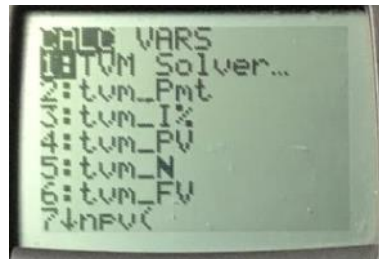
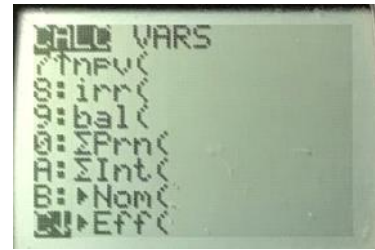
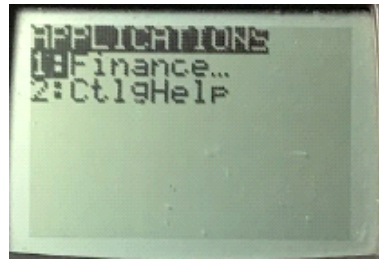
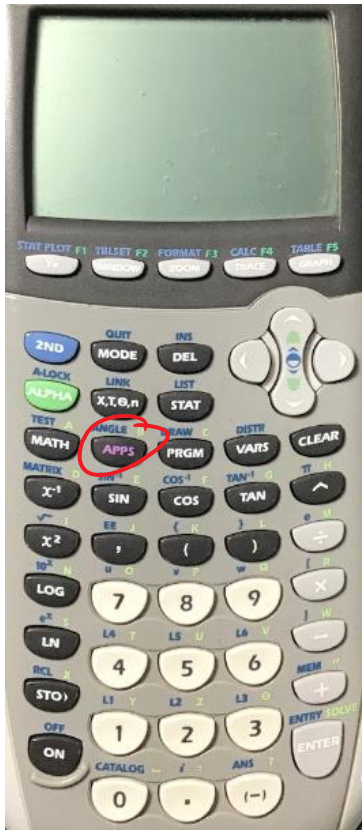
format mm.ddyy

② enter first date DT1 1.3119  
press down arrow

③ enter 2<sup>nd</sup> date DT2 4.3019  
press down arrow

④ for DED press CPT → 84 days.

Finance Menu: dbd



*mm.ddyy*  
*dbd(1.3119, 4.3019) enter*

## Section 1.5: Compound interest

**Compound interest** is when the interest is automatically reinvested to earn additional interest.

$t$	$a(t)$
0	1
1	$1 + i$
2	$\underline{(1+i)} + \underline{i(1+i)} = (1+i) [1+i] = (1+i)^2$
3	$(1+i)^2 + i(1+i)^2 = (1+i)^2 [1+i] = (1+i)^3$
⋮	
n	$(1+i)^n$

The accumulation function for  $t = 1, 2, 3, \dots$  is  $a(t) = (1+i)^n$

where  $i$  is the compound interest rate for the period. }  $\rightarrow$  effective rate per period.

$$a(t) = (1+i)^t$$

Note: A constant rate of compound interest implies a constant effective rate of interest.

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = \frac{(1+i)^{n-1} [1+i-1]}{(1+i)^{n-1}} = i$$

Note: Unless otherwise stated, assume that interest is accrued over fractional period for compound interest according to the formula.

Example: Find the accumulated value of \$2,000 at the end of 2 years and 3 months invested at 6% per annum.

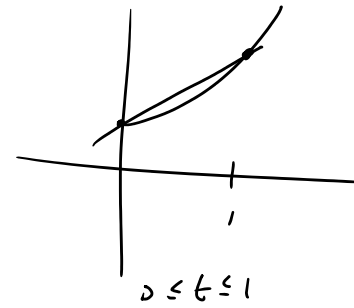
compound Rate per yr.

$$2 \text{ yr } 3 \text{ months} = 2 + \frac{3}{12} = 2 + \frac{1}{4} = 2.25 \text{ yrs.}$$

$$2000(1+.06)^{2.25} = 2000(1.06)^{2.25} = 2280.18$$

Example: Find the accumulated value of \$2,000 at the end of 2 years and 3 months invested at 6% per annum. Assume a simple interest during the final fractional period. Compare this the answer in the previous example.

$$\underbrace{2000 (1.06)^2}_{\substack{\text{represents} \\ \text{Balance at end} \\ \text{of 2 yrs.}}} \left[ \underbrace{1 + (.06)\left(\frac{3}{12}\right)}_{\substack{\text{Simple Interest} \\ \text{for last 3 months.}}} \right] = \$ 22\,80.91$$



Simple interest is used mostly for transactions less than one year.

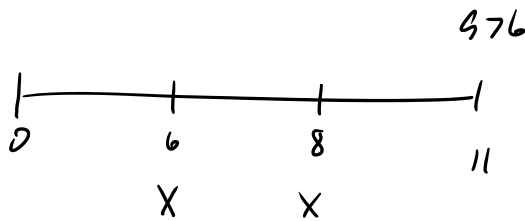
Compound interest is used almost exclusively for financial transactions covering a period of one year or more.

From this point assume compound interest in transactions over 1 year unless told otherwise.



Example: A deposit of \$X is invested at time 6 years at an annual effective interest rate of 8%. A second deposit of \$X is invested at time 8 years at the same interest rate. At time 11 years, the accumulated amount of the investment is \$976. Calculate X.

$$i = 8\%$$



$$X(1.08)^5 + X(1.08)^3 = 976$$

*an equation of value*

$$X \left[ 1.08^5 + 1.08^3 \right] = 976$$

$$X = \frac{976}{(1.08^5 + 1.08^3)} = \$357.63$$