Section 1.6: present value

We have seen that an investment of 1 will accumulate to $1 + i$ at the end of one period.

account value	1		1+ i
Period	**0**	i	**1**

The term $\mathbf{1+i}$ is often called the **accumulation factor**, since it accumulates the value of an investment at the beginning of a period to its value at the end of the period.

Question: How much should be invested initially so that the balance will be 1 at the end?

account value	**X**		**1**
Period	**0**	i	**1**

$$X + Xi = 1$$

$$X(1+i) = 1$$

$$X = \frac{1}{1+i} = v$$

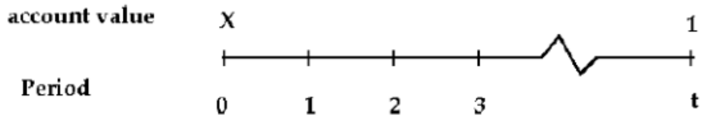
$$\left(\frac{1}{1+i}\right)(1+i) = 1$$

The term $v =$ $1/1+i$ is often called a **discount factor**, since it "discounts" the value of an investment at the end of a period to its value at the beginning.

$$X = \frac{1}{1+i} \cdot 1 = v \cdot 1$$

compound.

Find the amount that should be invested initially in order to accumulate an amount of 1 at the end of t periods.



$$X(1+i)^t = 1$$

$$X = \frac{1}{(1+i)^t} \cdot 1 = \left(\frac{1}{1+i}\right)^t 1 = \underbrace{v}^{t} \cdot 1$$

v^t is the present value of \$1 after t periods. (compound Interest)

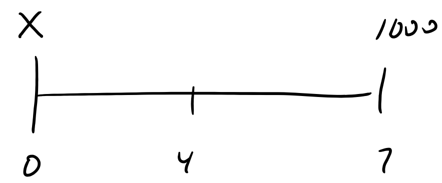
Note: The book uses $\underbrace{a^{-1}(t)}$ to denote the discount function, i.e. the reciprocal of $a(t)$. Should be $\underbrace{[a(t)]^{-1}}$.

For simple interest: discount function = $\frac{1}{1+it}$

Example: A payment of \$1000 is to be made at time 7 years. The annual effective rate is 6%.

(a) Determine the present value of this payment at time 0 and at time 4.
(b) How many years will it take the account to reach \$800, if the present value at time 0 is invested.

$i = 6\%$



$v = \frac{1}{1+i} = \frac{1}{1.06}$

time=0

Accumulation

$$X(1.06)^7 = 1000$$

$$X = \frac{1000}{(1.06)^7}$$

$$= \$665.06$$

discount

$$PV = X = v^7 \cdot 1000$$

$$= \left(\frac{1}{1.06}\right)^7 \cdot 1000$$

$$= 665.06 = PV_7$$

time 4)

$$PV_3 = 1000\,v^3 = 1000\left(\frac{1}{1.06}\right)^3 = 839.62$$

B)

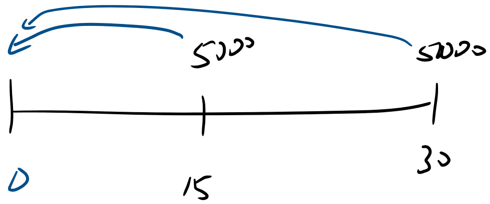
$$665.06(1.06)^X = 800$$

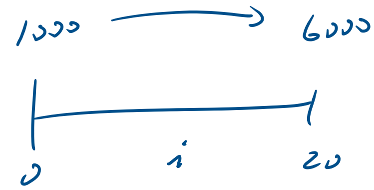
$$(1.06)^x = \frac{800}{665.06}$$

$$x \ln(1.06) = \ln\left(\frac{800}{665.06}\right)$$

$$X = \frac{\ln\left(\frac{800}{665.06}\right)}{\ln(1.06)} = 3.17 \text{ yrs.}$$

Example: An investment of \$1,000 will grow to \$6,000 after 20 years. Find the sum of the present values of two payments\$5,000 each which will occur at the end of the 15 and 30 years, assuming the same interest rate.





need to find annual eff. Rate.

$$1000(1+i)^{20} = 6000$$

$$(1+i)^{20} = 6$$

$$(1+i) = 6^{1/20}$$

$$\text{Answer} = 5000v^{15} + 5000v^{30}$$

$$v = \frac{1}{1+i} = \frac{1}{6^{1/20}}$$

$$= 5000\left(\frac{1}{6^{15/20}}\right) + 5000\left(\frac{1}{6^{30/20}}\right)$$

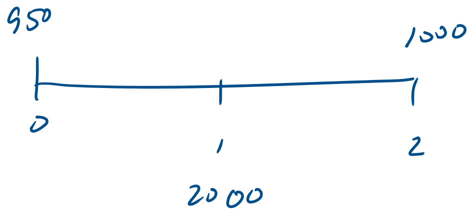
$$= \frac{5000}{6^{15/20}} + \frac{5000}{6^{30/20}} = 1644.44$$

Example: Mark owes you some money. He has given you two options.

Option 1: He will make a payment of \$950 now and then another payment of \$1000 at time 2.

Option 2: He will give you only one payment of 2000 at time 1.

What annual effective interest rate would make both options equivalent?



$V = \frac{1}{1+i}$

$1+i = V^{-1}$

$\underline{t=1}$

$950(1+i) + 1000V = 2000$

$\underline{\text{at } t=2}$

$950(1+i)^2 + 1000 = 2000(1+i)$

$\underline{\text{at } t=0}$

$950 + 1000V^2 = 2000V$

$1000V^2 - 2000V + 950 = 0$

$V = 1.223606798$ or $V = .77639$

$1+i = (1.223606798)^{-1}$

$1+i = .817256$

$i = \text{negative \#}$

$1+i = (.77639)^{-1}$

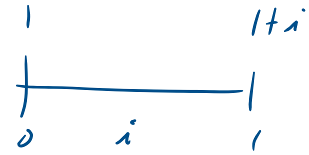
$1+i = 1.288007156$

$i = .288007156$

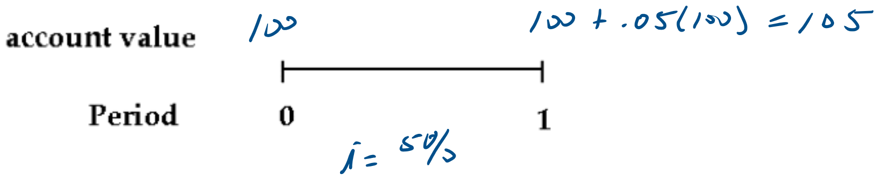
28.8007%

annual eff.
Rate is 28.8007%

Section 1.7: Effect rate of discount



• If Sue borrows \$100 from a bank for 1 year at an effective rate of interest of 5%, then at the end of one period(one year), Sue would pay back the original loan of \$100 plus interest of \$5 or a total of \$105.

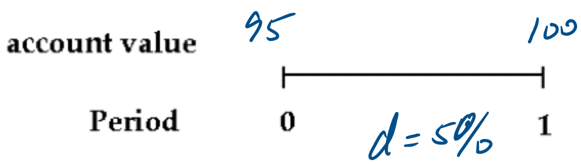


• If Bob borrows \$100 for one year at an effective rate of discount of 5%, then the bank will collects its interest of 5%, or \$5, in advance and will give Bob only \$95. At the end of the period, Bob will repay \$100.

$$95(1+i) = 100$$

$$\vdots$$

$$i = 5.26316\%$$



The effective rate of interest, i, is a measure of the interest paid at the end of the period. Or the ratio of the interest earned in the period, to the amount invested at the begining of the period.

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)} \text{ for } n = 1, 2, 3, \cdots$$

The **effective rate of discount**, d, is a measure of interest paid at the beginning of the period. Or the ratio of the amount of interest (amount of discount or just discount) earned during the period to the amount invested at the end on the period.

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{I_n}{A(n)} \text{ for } n = 1, 2, 3, \cdots$$

Note: The effective rate of discount is constant for each period when compounding.

Compound Interest

Assume a discount of d each period.

account value				$(1-d)^3$	$(1-d)^2$	**1–d**	**1**
Period	**0**	**1**	...	**n–3**	**n–2**	**n–1**	**n**

at n-2) $(1-d) - d(1-d) = (1-d)(1-d) = (1-d)^2$

dIscount t periods

$(1-d)^t = v^t$

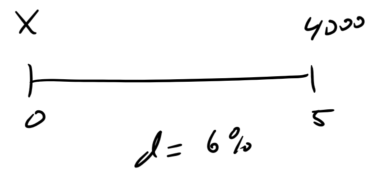
$\boxed{1-d = v}$

$1-d = \frac{1}{1+i}$

$(1+i)(1-d) = 1$

For an annual compound rate of discount, d:

- The present value of a payment of \$1 to be made in t years is $(1-d)^t$
- The accumulated value after t years of a deposit of \$1 is $(1-d)^{-t}$



Example: How much should an investor deposit today to have \$4,000 in 5 years if the annual rate of discount is 6%?

$$4000(1-d)^5 = 4000(1-.06)^5 = 2935.62$$

$$4000\,v^5 = 4000(.94)^5$$

$$1-d = v =$$

$$v = .94$$

Example: Compare the accumulated amount of \$1000 invested for 10 years at an annual rate of interest of 6% verses an annual rate of discount of 6%.

Interest

$$1000(1.06)^{10} = 1790.85$$

discount

$$1000(1-.06)^{-10} = 1856.61$$

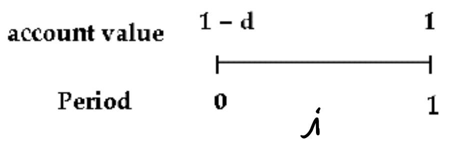
diff. is \$74.76

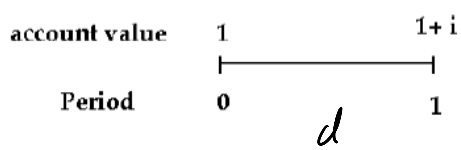
Pg 5: relationship of d and i

The relationship between d and i

Note: The following relationships are only valid for compound discount/compound interest and not for simple discount/simple interest (unless the number of periods is one).

Concept of equivalency: Two rates of interest or discounts are said to be equivalent if a given amount of principal invested for the same length of time at each of the rates produces the same accumulated value.





$$i_n = \frac{I_n}{A(n-1)} = \boxed{i = \frac{d}{1-d}}$$

$$d_n = \frac{I_n}{A(n)} = \boxed{d = \frac{i}{1+i}} = iv$$

$$\boxed{(1-d)(1+i)=1}$$

Simple Discount:

The amount of discount earned each period is a constant. For an annual simple rate of discount, d:

- The present value of a payment of $1 to be made in t years is $1-dt$
- The accumulated value after t years of a deposit of $1 is $(1-dt)^{-1}$

Note: simple discount is generally only used for terms less than 1 year.

Example: What is the present value of $1000 due in 10 days at a simple daily discount rate of 10%?

$$PV = 1000\left[1-.10(10)\right] = 0$$

Example: An investment of \$10,000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 2 years. Compute the effective rate of discount for the second year.

t	$A(t)$	I_n	i_n
0	10,000		
1	10,600	$I_1 = 600$	$i_1 = 6\%$
2	11,024	$I_2 = 424$	$i_2 = 4\%$

$$d_2 = \frac{I_2}{A(2)} = \frac{424}{11024} = 3.846\%$$

$$(1-d)(1+i) = 1$$

$$(1-d)(1.04) = 1$$

$$d_2 = \frac{i_2}{1+i_2} = \frac{.04}{1.04} = 3.846\%$$

Example: Find the accumulated value of $1000 at the end of 7 years and 5 months invested at an effective rate of discount of 4% assuming simple discount in the fractional period.

annual.

$$1000\,(1-.04)^{-7}\left[1-.04\left(\frac{5}{12}\right)\right]^{-1}$$