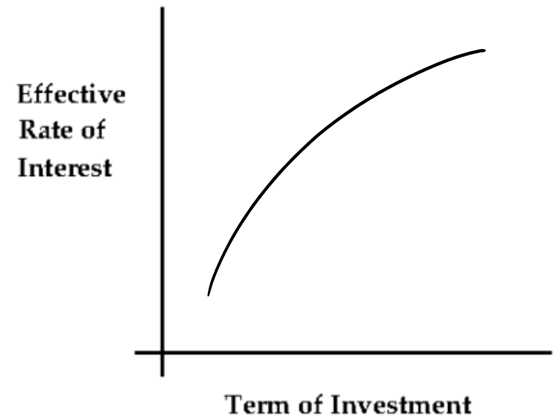
Section 10.2: Yield Curves

Term structure of interest refers to the phenomenon in which rates of interest differ depending on the term of otherwise identical financial instruments.

A **yield curve** is a graph that displays the relationship between rates of interest and the term of investment.

A **normal yield curve** is one that has a positive slope. i.e. increases with the length of the investment.



So why is this curve considered normal?

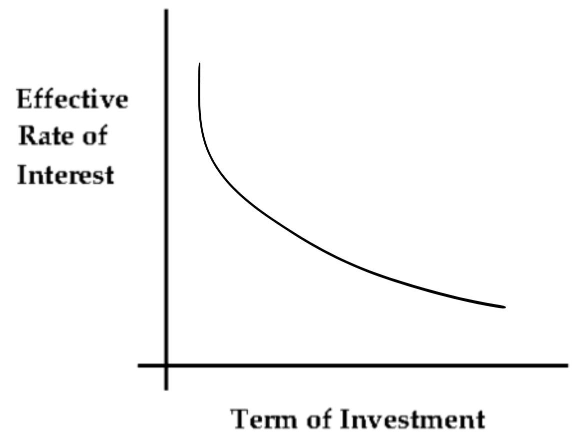
Expectations Theory: A higher percentage of individuals and business firms have an expectation that rates will rise in the future than the percentage which expect them to fall.

Liquidity Preference Theory: Assumes that individuals and firms prefer to invest for short periods so that they will have early access to their funds. A higher rate of interest with a longer term is an incentive to induce investors to commit their funds for longer times.

Inflation Premium Theory: Assumes investors demand higher rates of interest on longer investments to protect against the uncertainty of future rates of inflation.

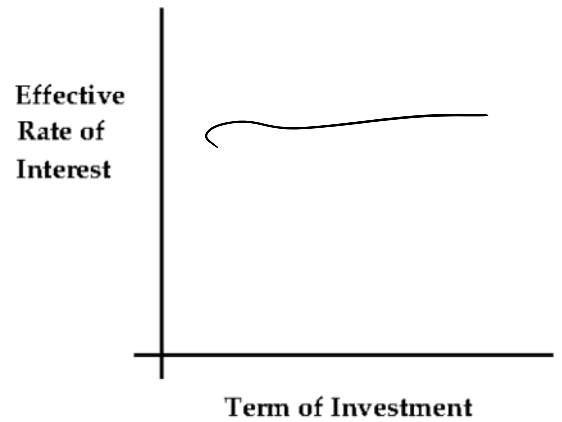
An **Inverted Yield Curve**.

In the bond market, short-term interest rates are heavily influenced by the policies of the Federal Reserve Board which may be setting high short-term rates to fight current inflation. Long-terms rates are determined by supply and demand in the bond market and may be lower due to expectations of lower inflation rates in the future.



A **Flat Yield Curve**.

May occur in periods of stability in which investors do not expect dramatic changes in the economy, investment rates, or future inflation rates.



The most basic yield curve is determined by the yields on zero coupon bonds of carrying terms backed by the US Treasury. A yield curve can also be constructed based on corporate bonds. Such a curve would lie above the Treasury yield curve since yields on corporate bonds are greater than treasury bonds.

Section 10.3: Spot Rates

The interest rates on the yield curve are often called **spot rates**.

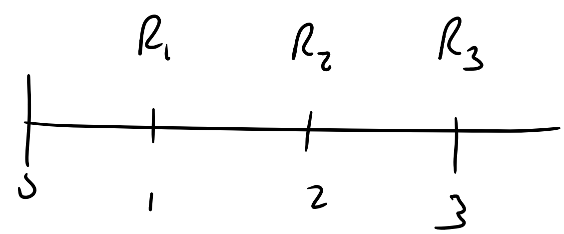
s_t = spot rate for a term of length t, expressed as an annual effective rate for any value of t.

In section 7.2 the net present value formula is based on a single rate of interest i.

$$NPV = P(i) = \sum_{t=0}^{n} R_t v^t$$

When considering the term structure of interest rates, the net present value formula can be generalized using spot rates. $P(s)$ denotes the fact that the net present value is based on a series of spot rates s_t.

$$NPV = P(s) = \sum_{t=0}^{n} (1 + s_t)^{-t} R_t$$



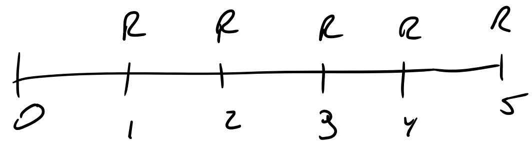
$$PV = R_1(1+s_1)^{-1} + R_2(1+s_2)^{-2} + R_3(1+s_3)^{-3}$$

Example: You are given the following selected values from a yield curve.

Term	1 year	2 years	3 years	4 years	5 years
Spot rate(annual effective)	7.00%	8.00%	8.75%	9.25%	9.50%

$R = 1000$

(a) Find the present value of payments of \$1000 at the end of each year for 5 years using these spot rates.



$$PV = 1000(1.07)^{-1} + 1000(1.08)^{-2} + 1000(1.0875)^{-3} + 1000(1.0925)^{-4} + 1000(1.095)^{-5}$$

$$= 1000\left[(1.07)^{-1} + 1.08^{-2} + 1.0875^{-3} + 1.0925^{-4} + 1.095^{-5}\right] = 3906.63$$

(b) What level yield rate would produce an equivalent value?

$$3906.63 = 1000\, a_{\overline{5}|i}$$

$$i = 8.83235\% \text{ annual eff.}$$

Example: You are given the following selected values from a yield curve.

maturity time (in half years)	1	2	3	4
		1yr.		2yrs
Spot rate(nominal rate convertible semiannually)	4%	5%	6%	7%

Find the present value of payments of $500 at the end of each semiannual period for 2 years using these sport rates.

$$PV = 500\left(1+\frac{.04}{2}\right)^{-2\left(\frac{1}{2}\right)} + 500\left(1+\frac{.05}{2}\right)^{-2(1)} + \cdots$$

$$PV = 500(1.02)^{-1} + 500(1.025)^{-2} + 500(1.03)^{-3} + 500(1.035)^{-4}$$

$$= 1859.395$$

Section 10.4: Relationship with Bond Values

The formula for the price of a bond was computed assuming a level yield rate(yield to maturity).

$$\underbrace{P = Fra_{\overline{n}|i} + Cv_i^n}$$

n coupons.

The formula can be generalized with spot rates.

$$P = Fr\sum_{i=1}^{n}(1+s_i)^{-i} + C(1+s_n)^{-n}$$

How does the price of the bond computed by these two methods compare?

According to the modern finance theory, these two prices must be equal. (Law of one Price)

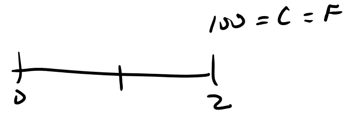
The idea is that any coupon bond may decomposed into a series of zero coupon bonds, each of which can be valued precisely using its associated spot rate.

Example: Suppose that the current term structure has the following spot rates (annual effective):

term	0.5 year	1 year	1.5 year	2 years
spot rate(annual effective)	8%	9%	10%	11%

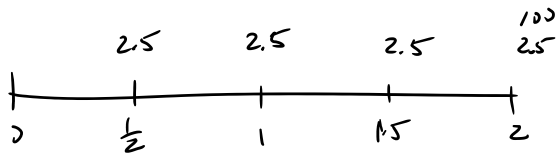
Find the price of a 2-year \$100-par value bond with

(A) no coupons (zero coupon bond)



$$\text{price} = 100(1.11)^{-2}$$
$$= 81.16224$$

(B) 5% semiannual coupons.



$r = 2.5\%$

$F = 100 \qquad Fr = 2.5$

$$PV = \text{price} = 2.5(1.08)^{-1/2} + 2.5(1.09)^{-1} + 2.5(1.10)^{-1.5} + 102.5(1.11)^{-2}$$
$$= 90.05746$$

(C) 10% semiannual coupons

$r = 5\% \qquad Fr = 5$

$$\text{price} = 5\left[1.08^{-1/2} + 1.09^{-1} + 1.10^{-1.5}\right] + 105(1.11)^{-2}$$
$$= 98.95268$$

$\rightarrow r$

At-par yield is the yield rate, based on spot rates, that would cause a bond to have a yield rate equal to its modified coupon rate. Meaning the bond would sell at par.

price = F = C

Example: Suppose that the current term structure has the following spot rates (annual effective):

term	0.5 year	1 year	1.5 year	2 years
spot rate(annual effective)	8%	9%	10%	11%

For a 2-year $100-par value bond with semiannual coupons, find the at-par yield rate for the bond.

↳ find r ⟹ price = 100

$$100 = 100r\left[(1.08)^{-1/2} + (1.09)^{-1} + (1.10)^{-1.5} + (1.11)^{-2}\right] + 100(1.11)^{-2}$$

$$1 = r\left[\;\right] + (1.11)^{-2}$$

$$\frac{1-(1.11)^{-2}}{\left[\;\right]} = r \longrightarrow 5.2943478\%$$

↑ Semiannual eff. Rate

$\longrightarrow r = 10.5887\%$

nominal semiannual Rate

It is possible to determine a set of spot rates given a set of coupon bond prices at each of the durations for which a spot rate is to be determined. This method recursive and is sometimes called the **bootstrap method**.

Let P_t be the price of a t−year coupon bond.

One yr coupon bond: $P_1 = (Fr + C)(1+S_1)^{-1} = \frac{Fr + C}{1+S_1}$

2yr coupon bond: $P_2 = \frac{Fr}{1+S_1} + \frac{Fr + C}{(1+S_2)^2}$

3yr coupon bond: $P_3 = \frac{Fr}{1+S_1} + \frac{Fr}{(1+S_2)^2} + \frac{Fr + C}{(1+S_3)^3}$

Example: The following table has the prices of $1000 par value bonds with 10% annual coupons.

term	1 year	2 year	3 year
price	1028.04	1036.53	1034.47

$Fr = 10$

Find the spot rates for $t = 1, 2, 3$ that are implied by these bond prices.

for 1yr

$$1028.04 = \frac{10 + 1000}{1+S_1} \longrightarrow S_1 = 7\%$$

for 2yr

$$1036.53 = \frac{10}{1+S_1} + \frac{10+1000}{(1+S_2)^2} \longrightarrow S_2 = 8\%$$

for 3yr

$$1034.47 = \frac{10}{1+S_1} + \frac{10}{(1+S_2)^2} + \frac{10+1000}{(1+S_3)^3} \longrightarrow S_3 = 8.75\%$$

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prices of par value 100 bond Annual coupons

term	coupon	price per 100 of par	Fr
1yr	10%	106.7961	10
2yr	2%	94.4588	2
3yr	8%	100.7571	8

find S_1, S_2, S_3 (annual eff) spot Rates.

1yr bond $$106.7961 = \frac{10+100}{1+S_1} \longrightarrow S_1 = 3\%$$

2yr Bond $$94.4588 = \frac{2}{1+S_1} + \frac{2+100}{(1+S_2)^2} \longrightarrow S_2 = 5\%$$

3yr Bond $$100.7571 = \frac{8}{1+S_1} + \frac{8}{(1+S_2)^2} + \frac{8+100}{(1+S_3)^3} \longrightarrow S_3 = 8\%$$

now find the price of a 3yr 12% annual coupon bond with F=100

$$\text{price} = 12(1.03)^{-1} + 12(1.05)^{-2} + 112(1.08)^{-3} = 111.444$$