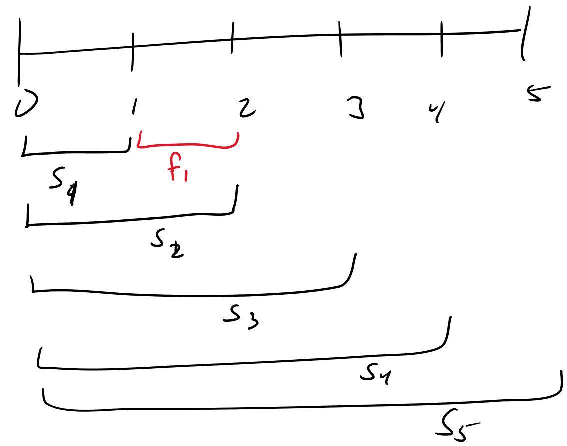
Section 10.5: Forward Rates

A **forward rate** is an expected spot rate which will come into play in the future.



Consider the following: A firm needs to borrow money for two years. The one-year spot rate is 7% and the two-year spot rate is 8%.

The firm has two options: (a) borrow all the money at the two-year spot rate or (b) borrow for one year at the one-year spot rate and then borrow for the second year at the one-year spot rate in effect a year later. The second one-year spot rate is called a **forward rate**.

A set of spot rates will imply a set of forward rates.

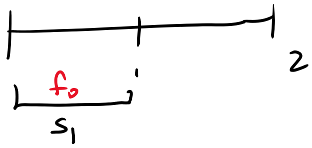
Unless told otherwise, forward rates are quoted as annual effective rates.

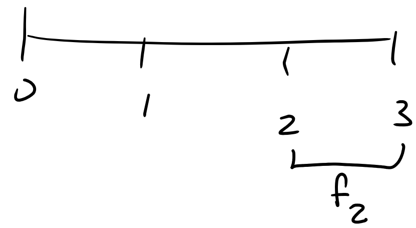
Notation:

S_p = 1yr spot Rate. Starts at $t=0$

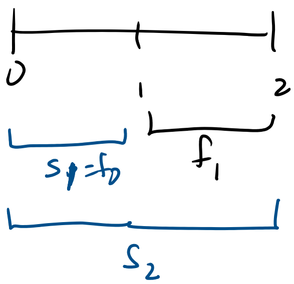
- s_t is the spot rate from time 0(year 0) to time t (year t).
- f_t is the one year forward rate from year t to year $t+1$.
  i.e. f_2 means, starting 2 years from now the effective rate of interest for one year will be f_2.

Interpret f_0. $= s_1$





f_1 = starting 1yr from now ($t=0$) this is the 1yr Rate

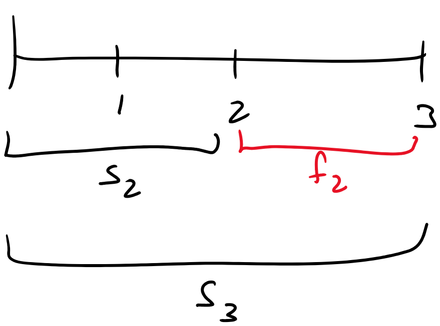


$$1(1+s_2)^2 = 1(1+s_1)(1+f_1)$$

$$1+f_1 = \frac{(1+s_2)^2}{1+s_1}$$

$$f_1 = \frac{(1+s_2)^2}{1+s_1} - 1$$

f_2 is the 1yrs forward Rate starting at $t=2$



$$(1+s_3)^3 = (1+s_2)^2(1+f_2)$$

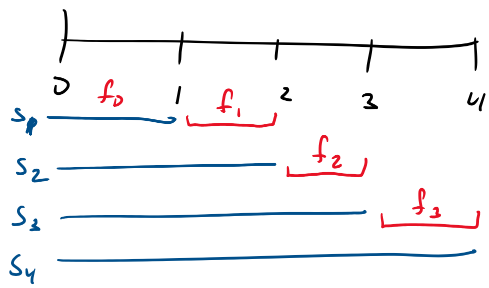
$$\boxed{1+f_2 = \frac{(1+s_3)^3}{(1+s_2)^2}}$$

$$f_{n-1} = \frac{(1+S_n)^n}{(1+S_{n-1})^{n-1}} - 1$$

general formula for forward Rates

Example: Given the following spot rates, find all one-year forward rates that can be determined from this information.

term	1 year	2 year	3 year	4 year
Spot rate, s_t	6%	6.25%	7%	7.5%
	s_1	s_2	s_3	s_4



$$f_0 = s_1 = 6\%$$

f_1

$$(1+s_2)^2 = (1+s_1)(1+f_1)$$

$$1+f_1 = \frac{(1+s_2)^2}{1+s_1} = \frac{(1.0625)^2}{1.06} \longrightarrow f_1 = 6.5005896\%$$

f_2

$$(1+s_3)^3 = (1+s_2)^2(1+f_2)$$

$$(1+f_2) = \frac{(1+s_3)^3}{(1+s_2)^2} = \frac{(1.07)^3}{(1.0625)^2} \longrightarrow f_2 = 8.5159\%$$

f_3

$$(1+f_3) = \frac{(1+s_4)^4}{(1+s_3)^3} = \frac{(1.075)^4}{(1.07)^3} \longrightarrow f_3 = 9.014\%$$

$$(1+f_0)(1+f_1)(1+f_2) = \underline{1.225043}$$

$$(1.07)^3 = \underline{1.225043}$$

In general, an -n-year spot rate can be expressed in terms of a set of n one-year forward rates.

$$(1+s_3)^3 = (1+f_0)(1+f_1)(1+f_2)$$

Example: The following table has the prices of \$1000 par value bonds with 10% annual coupons.

term	1 year	2 year	3 year
price	1028.04	1036.53	1034.47

$Fr = 1000(.10) = 100$

Find the forward rates for $t = 0, 1, 2$ that are implied by these bond prices.

1yr)

$$1028.04 = \frac{\overbrace{100 + 1000}^{Fr + C}}{1+f_0} \longrightarrow 1 + f_0 = \frac{1100}{1028.04} \longrightarrow f_0 = 7\%$$

2yr)

$$1036.53 = \frac{100}{1+f_0} + \frac{100 + 1000}{(1+f_0)(1+f_1)} \longrightarrow f_1 = 9.009\%$$

note $(1+s_2)^2 = (1+f_0)(1+f_1)$

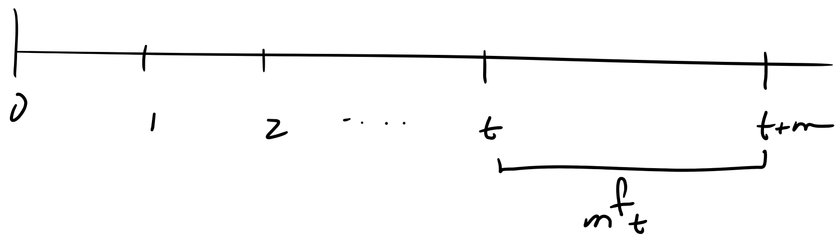
3yr)

$$1034.47 = \frac{100}{1+f_0} + \frac{100}{(1+f_0)(1+f_1)} + \frac{100 + 1000}{(1+f_0)(1+f_1)(1+f_2)} \longrightarrow f_2 = 10.27\%$$

$f_2 = {}_1f_2$

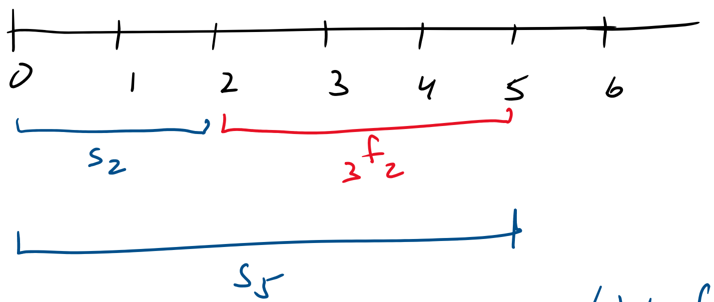
Forward rate over m-years

The m-year forward rate (annual effective) which applies over the period from time t to time $t+m$ is denoted by ${}_mf_t$ or $f_{t,t+m}$



Example: Given the following spot rates, compute the forward rate that is applicable for 3 years starting 2 years from now. $\rightarrow {}_3f_2$

term	1 year	2 year	3 year	4 year	5 year
Spot rate, s_t	6%	6.25%	7%	7.5%	8%



$$(1+s_5)^5 = (1+s_2)^2\,(1+{}_3f_2)^3$$

$$(1+{}_3f_2)^3 = \frac{(1+s_5)^5}{(1+s_2)^2}$$

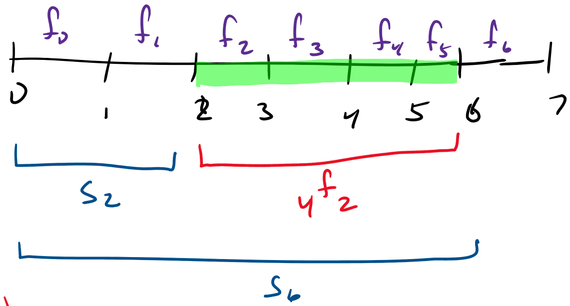
$$1+{}_3f_2 = \left(\frac{(1+s_5)^5}{(1+s_2)^2}\right)^{1/3}$$

$${}_3f_2 = \left(\frac{(1.08)^5}{(1.0625)^2}\right)^{1/3} - 1 \quad \Rightarrow \quad {}_3f_2 = 9.18265\%$$

Example: Consider the forward rates given below.

t	0	1	2	3	4	5	6
f_t	2%	4%	5%	7%	8%	9%	3%

Compute ${}_4f_2$.



$$(1+S_6)^6 = (1+f_0)(1+f_1)(1+f_2)(1+f_3)(1+f_4)(1+f_5)$$

$$(1+S_2)^2 = (1+f_0)(1+f_1)$$

$$(1+S_6)^6 = (1+S_2)^2(1+{}_4f_2)^4$$

$$(1+{}_4f_2)^4 = \frac{(1+S_6)^6}{(1+S_2)^2}$$

$$(1+{}_4f_2)^4 = (1+f_2)(1+f_3)(1+f_4)(1+f_5)$$

$\vdots$

$${}_4f_2 = 7.23976\%$$

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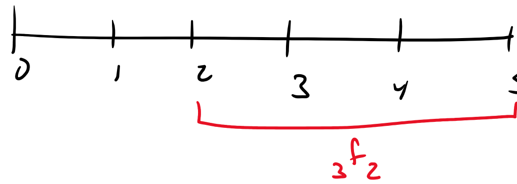
prices of 5 zero coupon bonds

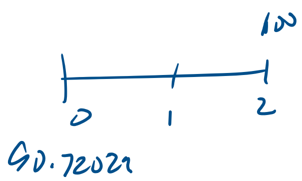
Term	1yr	2yr	3yr	4yr	5yr
price per 100 par	97.0874	90.72029	81.6298	73.5030	64.2529

determine ${}_3f_2$



need S_2 + S_5

for S_2



$$90.72029 = \frac{100}{(1+S_2)^2}$$

$$(1+S_2)^2 = \frac{100}{90.72029} \longrightarrow S_2 = 4.99\%$$

for S_5

$$64.2529 = \frac{100}{(1+S_5)^5} \longrightarrow S_5 = 9.25\%$$

$$(1+S_5)^5 = (1+S_2)^2(1+{}_3f_2)^3 \longrightarrow {}_3f_2 = 12.1856\%$$