Section 11.2: Duration

The goal of this section is to develop indices to measure the timing of future cash flows.

The **term to maturity** distinguishes a 10 year bond from a 20 year bond, but would not distinguish between two 10-year bonds, one with 5% coupons and the other with 10% coupons.

The **method of equated time**, a better index, is computed as the weighted average of the various times of payments, where the weights are the various amounts paid. (from section 2.4)

$$\bar{t} = \frac{\sum t * R_t}{\sum R_t}$$

10- year bond with 5% annual coupons: 100 par value

$$\bar{t} = \frac{1*5+2*5+3*5+\cdots+10*5+10*100}{10*5+100} = \frac{5\left(\sum_{t=1}^{10} t\right)+10*100}{150} = \underline{8.50 \text{ yrs}}$$

10-year bond with 10% annual coupons:

$$\bar{t} = \frac{1*10+2*10+3*10+\cdots+10*10+10*100}{10*10+100} = \frac{10\left(\sum_{t=1}^{10} t\right)+10*100}{200} = \underline{7.75 \text{ yrs}}$$

Understanding the Macaulay Duration

The metric is named after its creator, Frederick Macaulay. Macaulay duration can be viewed as the economic balance point of a group of cash flows. Another way to interpret the statistic is that it is the weighted average number of years that an investor must maintain a position in the bond until the present value of the bond's cash flows equals the amount paid for the bond.

Macaulay duration or simply just **duration** (MacD), an even better index, is a weighted average of various times of payments with the present value of each cash flow is used as the weight. Units are in years.

$PV_t = v^t R_t$

$$\overline{d} = \frac{\sum_{t=1}^{n} t * v^t R_t}{\sum_{t=1}^{n} v^t R_t} = \sum_{t=1}^{n} t\left(\frac{PV_t}{price}\right)$$

$Price = \sum v^t R_t$

- This method assumes the payment period and the interest conversion period coincide.
- The duration, $\overline{d}$ is a decreasing function of i. As i increases, the payments at later times are discounted more than with the smaller i, giving less weight to the later times.
- If there is only one future cash flow, then $\overline{d}$ is the point in time at which the cash flow is made.
- If the cash flow payments are equal, i.e. $R_t = R$, then duration formula may be expressed as the following.

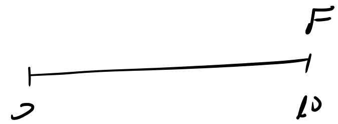
$$\overline{d} = \frac{\sum_{t=1}^{n} t * v^t R}{\sum_{t=1}^{n} v^t R} = \frac{R\sum_{t=1}^{n} t * v^t}{R\sum_{t=1}^{n} v^t} = \frac{\sum_{t=1}^{n} t * v^t}{\sum_{t=1}^{n} v^t} = \frac{(Ia)_{\overline{n|}}}{a_{\overline{n|}}} = \overline{d}$$

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$$

$$(Ia)_{\overline{n|}} = \overset{P}{1}\, a_{\overline{n|}} + \overset{Q}{1}\, \frac{a_{\overline{n|}} - nv^n}{i}$$

Example: Find the Macaulay duration of the following investments assuming the effective rate of interest is 8%.

(A) A ten-year zero coupon bond.



$$\bar{d} = \frac{10 F v^{10}}{F v^{10}} = 10 \text{ yrs.}$$

(B) A 10-year bond with 6% annual coupons. $F(.06) = \text{coupon}$

$$\bar{d} = \frac{\sum_{t=1}^{10} t \cdot F(.06) v^t + F \cdot 10 v^{10}}{F(.06) a_{\overline{10}|} + F v^{10}} = \frac{.06 \sum t v^t + 10 v^{10}}{.06\, a_{\overline{10}|} + v^{10}}$$

$$= \frac{(.06)(Ia)_{\overline{10}|} + 10 v^{10}}{.06\, a_{\overline{10}|} + v^{10}} = \frac{.06\left[a_{\overline{10}|} + \frac{a_{\overline{10}|} - 10 v^{10}}{i}\right] + 10 v^{10}}{.06\, a_{\overline{10}|} + v^{10}}$$

$i = 8\%$

$$\bar{d} = 7.615109 \text{ yrs}$$

$$\sum F r v^t + F v^{10} = F r \sum v^t + F v^{10}$$
$$= F r\, a_{\overline{10}|}$$

Example: Find the Macaulay duration of the following investments assuming the effective rate of interest is 8%.

(C) A 10-year mortgage repaid with level annual payments of principal and interest.

$$\bar{d} = \frac{\sum tRv^t}{\sum Rv^t} = \frac{R\sum_{t=1}^{10} t\,v^t}{R\sum_{t=1}^{10} v^t} = \frac{R\,(Ia)_{\overline{10|}}}{R\,a_{\overline{10|}}} = 4.87131 \text{ yrs.}$$

Example: Find the Macaulay duration of the following investments assuming the effective rate of interest is 8%.

(D) A preferred stock paying level annual dividend into perpetuity.

$D =$ dividend

$$\bar{d} = \frac{\sum_{t=1}^{\infty} t D v^t}{\sum_{t=1}^{\infty} D v^t} = \frac{D \sum_{t=1}^{\infty} t v^t}{D \sum_{t=1}^{\infty} v^t} = \frac{\sum_{t=1}^{\infty} t v^t}{\sum_{t=1}^{\infty} v^t}$$

$$\bar{d} = \frac{(Ia)_{\overline{\infty|}}}{a_{\overline{\infty|}}} = \frac{\frac{1}{i} + \frac{1}{i^2}}{\frac{1}{i}} = 1 + \frac{1}{i} = 1 + \frac{1}{.08} = 13.5_{\text{yrs}}$$

$$\boxed{(Ia)_{\overline{\infty|}} = \lim_{n \to \infty} \left(a_{\overline{n|}} + \frac{a_{\overline{n|}} - n v^n}{i} \right) = \frac{1}{i} + \frac{1}{i^2}}$$

Example: The Macaulay duration of a 10-year annuity-immediate with annual payments of \$1,000 is 5.6 years. Calculate the Macaulay duration of a 10-year annuity-immediate with annual payments of \$50,000. Assume both annuities have the same effective rate.

$\bar{d}$ = 5.6 yrs also.

$$\overline{d} = \frac{\sum_{t=1}^{n} t * v^t R}{\sum_{t=1}^{n} v^t R} = \frac{R \sum_{t=1}^{n} t * v^t}{R \sum_{t=1}^{n} v^t} = \frac{\sum_{t=1}^{n} t * v^t}{\sum_{t=1}^{n} v^t} = \frac{(Ia)_{\overline{n}|}}{a_{\overline{n}|}}$$

Interest Rate Sensitivity

Let $P(i) = \sum_{t=1}^{n} R_t v^t = \sum_{t=1}^{n} R_t (1+i)^{-t}$ be the present value of a set of future cash flows.

The relative rate of change of this present value is called **interest rate sensitivity** of a set of future cash flows.

Define the **volatility** or **modified duration** (ModD) of this present value of a cash flow as $\overline{v} = \dfrac{-P'(i)}{P(i)}$.
Thus $\overline{v}$ is a function of the interest rate i.

$\overline{v} = \dfrac{-P'(i)}{P(i)}$

- Assuming $R_t > 0$ then it can be shown that $P'(i) < 0$ and $P''(i) > 0$ thus $P(i)$ is decreasing concave up function.

$P'(i) = \sum_{t=1}^{n} -t R_t (1+i)^{-t-1}$

- $P'(i)$ measures the instantaneous rate of change of the present value of the cash flow with respect to changes in i.
  The units of $P'(i)$ are dollars per 100 basis points (1% = 100 basis points).
- The units of $\overline{v}$ are $\dfrac{\$/100 \text{ basis points}}{\$} = 1$ per 100 basis points.

$(1+i)^{-t-1} = (1+i)^{-(t+1)} = v^{t+1}$

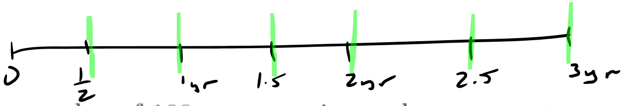
Relationship between $\overline{v}$ (ModD) and $\overline{d}$ (MacD

$$\overline{v} = \frac{-P'(i)}{P(i)} = \frac{-\sum_{t=1}^{n} -tR_t(1+i)^{-t-1}}{\sum_{t=1}^{n} R_t(1+i)^{-t}} = \frac{\sum_{t=1}^{n} tR_t v^{t+1}}{\sum_{t=1}^{n} R_t v^t} = \frac{\sum_{t=1}^{n} tR_t v^t * v}{\sum_{t=1}^{n} R_t v^t} = \frac{v * \sum_{t=1}^{n} tR_t v^t}{\sum_{t=1}^{n} R_t v^t} = v * \overline{d}$$

$$\overline{v} = \frac{\overline{d}}{1+i}$$

$\overline{v} = v\overline{d}$

$\overline{d} = \frac{1}{v}\overline{v} = (1+i)\overline{v}$



Example: A 3-year bond with a par value of 100, pays semiannual coupons at an annual rate of 8% and has a semiannual compound yield of 9%. The price of this bond is \$97.421.

Calculate the Macaulay duration and use it to determine the modified duration with respect to the semiannually compounded yield of the bond.

$r = 4\%$

$F = 100 = C$

coupon $= Fr = 4$

$i = 4.5\%$

found price by TVM solver

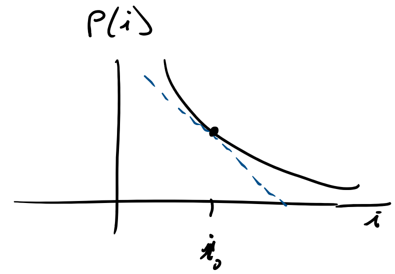
$$\bar{d} = \frac{\sum t R_t v^t}{\text{price}}$$

$$= \frac{1}{\text{price}}\Big[\left(\tfrac{1}{2}\right)(4)(1.045)^{-1} + 1(4)(1.045)^{-2} + (1.5)(4)(1.045)^{-3} + 2(4)(1.045)^{-4}$$

$$+ (2.5)(4)(1.045)^{-5} + 3(104)(1.045)^{-6}\Big]$$

$$\bar{d} = 2.7217 \text{ yrs}$$

$$\bar{\nu} = v\,\bar{d} = 2.6045 \text{ yrs.}$$



Modified Duration is an approximate measure of a bond's price sensitivity to changes in interest rates. If a bond has a duration of 6 years, for example, its price will rise about 6% if its yield drops by a percentage point (100 basis points), and its price will fall by about 6% if its yield rises by that amount.

Thus modified duration provides a method to estimate the change in the present value of a series of cash flows when the yield rate changes.

$$\bar{v} = -\frac{P'(i)}{P(i)} \implies P'(i) = -\bar{v}\,P(i)$$

Tangent line formula

$$\boxed{P(x) = P(a) + P'(a)(x-a)}$$

Tangent line approximation

$$P(i+h) \approx P(i) + P'(i)(\overset{i+h-i}{h})$$

$$= P(i) - \bar{v}\,P(i)\,h$$

$$= P(i)\left[1 - \bar{v}h\right]$$

Macaulay modified

$\bar{d}$ $\bar{v}$

Example: A bond with annual coupons has a price \$86.5798 when its annual yield is 8%. At this yield, the Macaulay duration is 7.61509. Estimate the price of the bond if the yield rises to 9%.

$P(8\%) = 86.5798$ $\bar{d} = 7.61509$ need $\bar{v} = \bar{d}v$

$= (7.61509)(1.08)^{-1}$

$= 7.051027578$

$h = 1\% = .01$

$P(i+h) \approx P(i)[1 - \bar{v}h]$

$\approx 86.5798[1 - (7.05103)(.01)]$

$P(9\%) \approx 80.475$

All of the above was based on the discrete yield rate i. If we consider a continuous force of interest δ then $P(\delta) = \sum_{t=1}^{n} e^{-\delta t} R_t$ and $P'(\delta) = \sum_{t=1}^{n} -te^{-\delta t} R_t$

$$\bar{d} = \frac{\sum_{t=1}^{n} t * v^t R_t}{\sum_{t=1}^{n} v^t R_t} = \frac{\sum_{t=1}^{n} t * e^{-\delta t} R_t}{\sum_{t=1}^{n} e^{-\delta t} R_t} = \frac{-P'(\delta)}{P(\delta)} = \overline{v}$$

Section 11.3: Convexity

Convexity is a measure of the curvature in the relationship between the present value of a set of cash flows and the yield of those cash flows.

$P(i)$

In particular for bonds, it is a relationship between the bond prices and bond yields that demonstrates how the duration of a bond changes as the interest rate changes. Convexity is used as a risk-management tool, which helps measure and manage the amount of market risk to which a portfolio of bonds is exposed.

The **convexity** of the present value of a set of cash flows is defined to be

$$\overline{c} = \frac{P''(i)}{P(i)} = \frac{\sum_{t=1}^{n} t(t+1)R_t v^{t+2}}{\sum_{t=1}^{n} R_t v^t}$$

$$P(i) = \sum_{t=1}^{n} R_t(1+i)^{-t} = \sum_{t=1}^{n} R_t v^t$$

$$P'(i) = \sum_{t=1}^{n} -tR_t(1+i)^{-t-1} = \sum_{t=1}^{n} -tR_t v^{t+1}$$

$$P''(i) = \sum_{t=1}^{n} t(t+1)R_t(1+i)^{-t-2} = \sum_{t=1}^{n} t(t+1)R_t v^{t+2}$$

Convexity in combination with modified duration provides another method to estimate the change in the present value of a series of cash flow when the yield rate changes.

$P(i+h) \approx P(i)\left[1 - \bar{v} h\right]$

Convexity in combination with modified duration provides another method to estimate the change in the present value of a series of cash flow when the yield rate changes.

2nd order Taylor polynomial.

$$P(i+h) \approx P(i) + P'(i)\,h + \frac{P''(i)}{2!}\,h^2$$

$$\approx P(i)\left\{1 + \frac{P'(i)}{P(i)}h + \frac{P''(i)}{P(i)}\frac{h^2}{2}\right\}$$

$$\approx P(i)\left\{1 - \bar{v}h + \bar{c}\frac{h^2}{2}\right\}$$

If we examine the rate of change of the modified duration, we notice the following.

$$\frac{d\overline{v}}{di} = \frac{d}{di}\frac{-P'(i)}{P(i)} = \frac{P(i) * -P''(i) + P'(i) * P'(i)}{(P(i))^2} = \frac{(P'(i))^2}{(P(i))^2} - \frac{P''(i)}{P(i)} = \overline{v}^2 - \overline{c}$$

$$= -\frac{P(i)\,P''(i)}{(P(i))^2} + \frac{(P'(i))^2}{(P(i))^2} =$$

Example: The current price of an annual coupon bond is \$100. The yield to maturity is an effective rate of 7% and $\frac{dP(i)}{di} = -650$.

$\frac{1}{v} = (1+i)$

(A) Calculate the $\underbrace{\text{Macaulay duration}}_{\bar{d}}$ of the bond.

$$\bar{d} = \frac{1}{v}\bar{v} = \frac{1}{v}\left(\frac{-P'(i)}{P(i)}\right) = (1.07)\cdot\left(-\frac{(-650)}{100}\right) = 6.955$$

$$\bar{v} = \frac{-(-650)}{100} = 6.5$$

(B) Using the given information, estimate the price of the bond when $i = 8\%$ instead of 7%.

$h = 1\% = .01$ (inc. by 100 basis points)

$P(i)[1 - \bar{v}h]$

$100[1 - (6.5)(.01)] =$

$$P(8\%) \approx P(7\%) + P'(7\%)\cdot h$$
$$= 100 + (-650)(.01) = 93.50$$

(C) Refine your price estimate by using both modified duration and convexity given that $\frac{d\bar{v}}{di} = -800$.

$$\frac{d\bar{v}}{di} = \bar{v}^2 - \bar{c} \implies \bar{c} = \bar{v}^2 - \frac{d\bar{v}}{di}$$
$$= (6.5)^2 - (-800)$$
$$= 842.25$$

$$P(8\%) \approx P(7\%)\left[1 - \bar{v}h + \bar{c}\frac{h^2}{2}\right]$$
$$= 100\left[1 - (6.5)(.01) + 842.25\cdot\frac{(.01)^2}{2}\right] = 97.71125$$

Computing convexity for a specific set of cash flows can be a daunting task in practice.

$$P(i) = \sum_{t=1}^{n} R_t(1+i)^{-t} = \sum_{t=1}^{n} R_t v^t$$

$$P'(i) = \sum_{t=1}^{n} -tR_t(1+i)^{-t-1} = \sum_{t=1}^{n} -tR_t v^{t+1}$$

$$P''(i) = \sum_{t=1}^{n} t(t+1)R_t(1+i)^{-t-2} = \sum_{t=1}^{n} t(t+1)R_t v^{t+2}$$

Example: You have a 15 year 1000 par value bond with an annual coupon rate of 7% has a yield of 5%. Compute Macaulay duration, modified duration, and the convexity of the bond.

If the yield on the bond drops by 50 basis points, approximate the price using both modified duration and convexity.

$h = -\frac{1}{2}\% = -.005$

par	1000	coupon rate	7.00%	annual rate	5.00%

t	R_t	v^t	t*v^t*R_t	v^t R_t	t v^(t+1) R_t	t(t+1)v^(t+2) R_t
1	70	0.95238	66.66667	66.66667	63.49206	120.93726
2	70	0.90703	126.98413	63.49206	120.93726	345.53504
3	70	0.86384	181.40590	60.46863	172.76752	658.16198
4	70	0.82270	230.35669	57.58917	219.38733	1044.70156
5	70	0.78353	274.23416	54.84683	261.17539	1492.43079
6	70	0.74622	313.41047	52.23508	298.48616	1989.90772
7	70	0.71068	348.23385	49.74769	331.65129	2526.86695
8	70	0.67684	379.03004	47.37876	360.98099	3094.12280
9	70	0.64461	406.10362	45.12262	386.76535	3683.47952
10	70	0.61391	429.73928	42.97393	409.27550	4287.64812
11	70	0.58468	450.20305	40.92755	428.76481	4900.16928
12	70	0.55684	467.74343	38.97862	445.46993	5515.34205
13	70	0.53032	482.59243	37.12249	459.61184	6128.15783
14	70	0.50507	494.96659	35.35476	471.39676	6734.23937
15	1070	0.48102	7720.32442	514.68829	7352.68993	112040.98938
		Total	12371.99473	1207.59316	11782.85212	154562.68965
				$P(i)$	$-P'(i)$	$P''(i)$

$$\bar{d} = \frac{12371.9947}{1207.59316} = 10.2452$$

$$\bar{v} = \frac{-P'(i)}{P(i)} = \frac{11782.85}{1207.59} = 9.7573$$

$$\bar{C} = \frac{P''(i)}{P(i)} = \frac{154562.6896}{1207.55} = 127.992353$$

$$\text{new price} \approx P(i)\left[1 - \bar{v}h + \bar{C}\frac{h^2}{2}\right]$$

$$= 1207.593\left[1 - \bar{v}(-.005) + \bar{C}\frac{(-.005)^2}{2}\right] = 1268.43945$$