# Section 3.2: Annuity-Immediate

An **annuity** may be defined as a series of payments made at equal intervals of time.

An **annuity-certain** is an annuity such that payments are certain to be made for a fixed period of time (the **term** of the annuity).

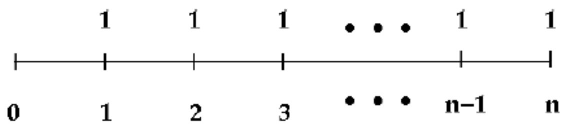
A **contingent annuity** is an annuity under which the payments are not certain. i.e. payments from a pension plan for the life of a retiree.

The interval between annuity payments is called the **payment period**. This chapter considers annuities where the payment period and the interest conversion period are equal and coincide.

## Section 3.2: Annuity-Immediate

An annuity under which payments are made at the end of each payment period for $n$ periods, where $n$ is a positive integer, is called an **annuity-immediate** or an **ordinary annuity** or just an **annuity**.

Consider the annuity where payments of $1 are made at the end of the period for $n$ periods.




$i$ = effective Rate per period.

- The present value (PV) of the annuity is denoted by $a_{\overline{n}|}$ or $a_{\overline{n}|i}$ .

$$PV = a_{\overline{n}|i} = v + v^2 + v^3 + \cdots + v^{n-1} + v^n$$

$$- \quad v a_{\overline{n}|} = v^2 + v^3 + \cdots + v^{n-1} + v^n + v^{n+1}$$

$$a_{\overline{n}|} - v a_{\overline{n}|} = v - v^{n+1}$$

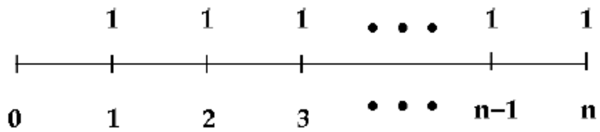
$$a_{\overline{n}|}(1-v) = v(1-v^n)$$

$$a_{\overline{n}|} = \frac{v(1-v^n)}{1-v} = \frac{v(1-v^n)}{vi}$$

$$\boxed{a_{\overline{n}|i} = \frac{1-v^n}{i}}$$

$$(1+i)v = 1$$
$$v + vi = 1$$
$$vi = 1 - v$$

Page 3: s angel n




$i$ is effective Rate per period.

- The accumulated value (FV) of the annuity is denoted by $s_{\overline{n}|}$ or $s_{\overline{n}|i}$.

t=1 t=2 t=n-2 t=n-1 t=n

$$FV = s_{\overline{n}|i} = 1(1+i)^{n-1} + 1(1+i)^{n-2} + \cdots + (1+i)^2 + (1+i) + 1$$

$$s_{\overline{n}|i} = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1}$$

$$-\quad (1+i)\, s_{\overline{n}|} = (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} + (1+i)^n$$

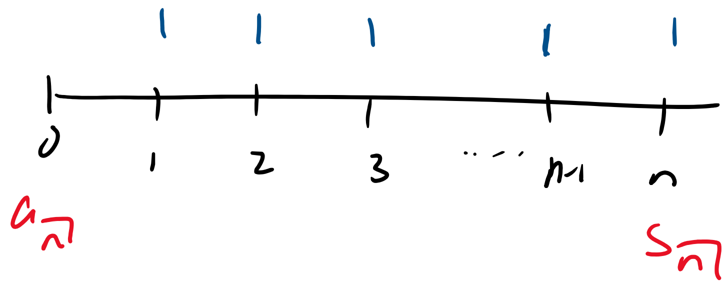
$$s_{\overline{n}|} - (1+i)\, s_{\overline{n}|} = 1 - (1+i)^n$$

$$s_{\overline{n}|}\left[1 - (1+i)\right] = 1 - (1+i)^n$$

$$s_{\overline{n}|} = \frac{1-(1+i)^n}{1-1-i} = \frac{1-(1+i)^n}{-i}$$

$$\boxed{s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}}$$

## Relationship between $s_{\overline{n}|}$ and $a_{\overline{n}|}$




$$a_{\overline{n}|}(1+i)^n = s_{\overline{n}|}$$

$$s_{\overline{n}|} v^n = a_{\overline{n}|}$$

## Geometric Progression/Geometric Series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k$$

$$s_{\overline{n}|} = S = a + ar + ar^2 + \cdots + ar^{n-1}$$

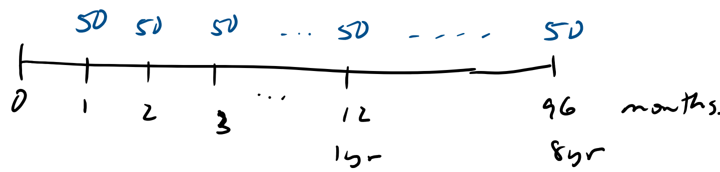
$$- \quad rS = \phantom{a + {}} ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$S - rS = a - ar^n$$

$$S = \frac{a(1-r^n)}{1-r}$$

Example: David will receive payments of \$50 at the end of each month for the next 8 years. Assume $i^{(12)} = 9\%$

(a) Find the present value of this annuity.




$$PV = 50 \cdot a_{\overline{96}|\frac{.09}{12}} = 50\left[\frac{1-v^{96}}{i}\right]$$

$$= 50\left[\frac{1-(1.0075)^{-96}}{.0075}\right]$$

$$= \$3412.92$$

find $i$ (eff. Rate per month)

$$\frac{i^{(12)}}{12} = \frac{.09}{12} = .0075$$

$$v = \frac{1}{1.0075} = (1.0075)^{-1}$$

$$v = (1+i)^{-1}$$

TVM solver

$a_{\overline{96}|.0075}$

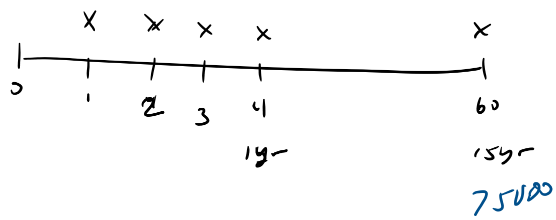
N = 96
I = .75 %
PV = solve
PMT = -1
FV = 0
P/Y = C/Y = 1
PMT: End.

(b) Find the accumulated value of this annuity.

$$FV = 50\, s_{\overline{96}|\,.0075} = 50\left[\frac{(1+.0075)^{96}-1}{.0075}\right] = 6992.81$$

$$3412.92\left(1+\frac{.09}{12}\right)^{96} =$$

Example: How much should be deposited at the end of each quarter so that at the end of 15 years the account balance is $75,000? Assume an annual effective rate of interest of 6.14%.




$$1+.0614 = \left(1+\frac{i^{(4)}}{4}\right)^4$$

$$\frac{i^{(4)}}{4} = (1.0614)^{1/4} - 1$$

$$= .0150087$$

by calc

$i = 6.14\% \to i^{(4)} = 6.00348\%$

$$75000 = X \, s_{\overline{60}|\frac{i^{(4)}}{4}}$$

$$75000 = X\left[\frac{(1+.0150087)^{60} - 1}{.0150087}\right]$$

$$X = \$779.28$$

TVM solver

N = 60

I = 1.50087%

PV = 0

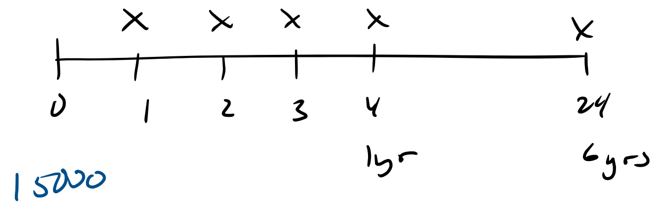
PMT = solve

FV = 75000

P/Y = C/Y = 1

PMT : end

Example: Bob invests a \$15,000 gift at nominal rate of 6% compounded quarterly. How much can be withdrawn at the end of every quarter to use up the fund exactly at the end of 6 years of college?




$i^{(4)} = 6\%$

$i = \frac{i^{(4)}}{4} = \frac{.06}{4} = .015$

Rate per quarter

$$15000 = X\, a_{\overline{24}|.015} = X \cdot \frac{1 - v^{24}}{i}$$

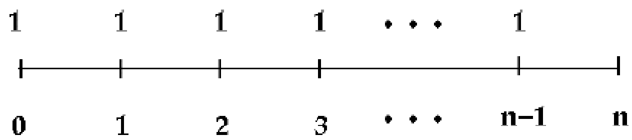
$.015 \to i$

$$v = \frac{1}{1.015}$$

$$X = \frac{15000\, i}{1 - v^{24}} = 748.86$$

Section 3.3: Annuity-Due

An **annuity-due** is an annuity for which payments are made at the beginning of the period.




- The present value (PV) of the annuity-due is denoted by $\ddot{a}_{\overline{n}|}$ or $\ddot{a}_{\overline{n}|i}$.

$$PV = \ddot{a}_{\overline{n}|i} = 1 + 1v + 1v^2 + \cdots + 1v^{n-1}$$

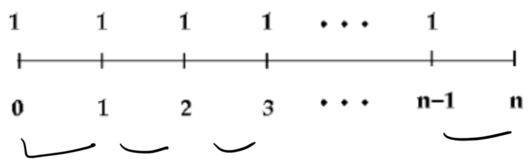
$v = 1-d$

$1 - v = d$

for a finite geometric series

$$\ddot{a}_{\overline{n}|i} = \frac{1(1-v^n)}{1-v} = \frac{1-v^n}{d}$$

pmt made at beginning of the period




at time n

- The accumulated value (FV) of the annuity-due is denoted by $\ddot{s}_{\overline{n}|}$ or $\ddot{s}_{\overline{n}|i}$.

$$FV = \ddot{s}_{\overline{n}|i} = 1(1+i) + 1(1+i)^2 + \cdots + (1+i)^n$$

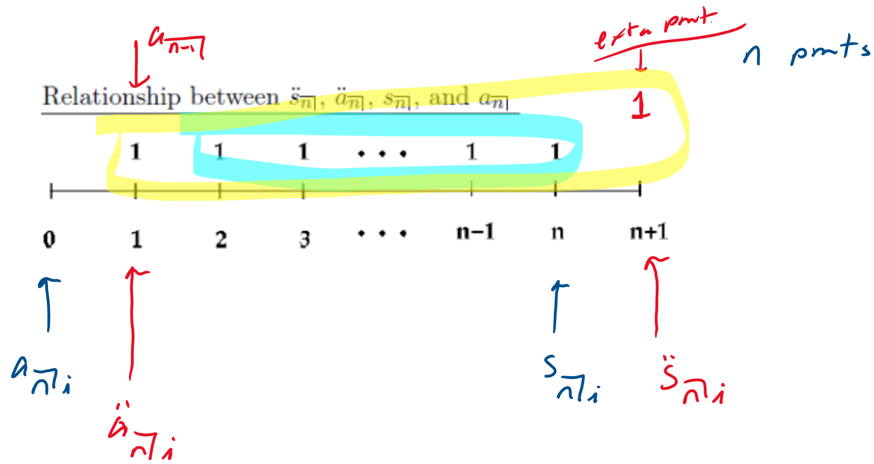
$$= (1+i)\left[1 + (1+i)^1 + (1+i)^2 + \cdots + (1+i)^{n-1}\right]$$

$$= (1+i)\left[\frac{1\left(1-(1+i)^n\right)}{1-(1+i)}\right]$$

$$= (1+i)\left[\frac{1-(1+i)^n}{-i}\right] = (1+i)\left[\frac{(1+i)^n-1}{i}\right]$$

$$\ddot{s}_{\overline{n}|i} = \frac{(1+i)^n-1}{\frac{i}{1+i}} = \frac{(1+i)^n-1}{d}$$

Relationship between $\ddot{s}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$, and $a_{\overline{n}|}$




True or False

$$\ddot{a}_{\overline{n}|} = a_{\overline{n-1}|} + 1$$

$$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

$$\ddot{a}_{\overline{n}|i} = a_{\overline{n}|i}(1+i)$$

$$\ddot{s}_{\overline{n}|} = s_{\overline{n}|}(1+i)$$

$$a_{\overline{n}|} = \ddot{a}_{\overline{n}|}v$$

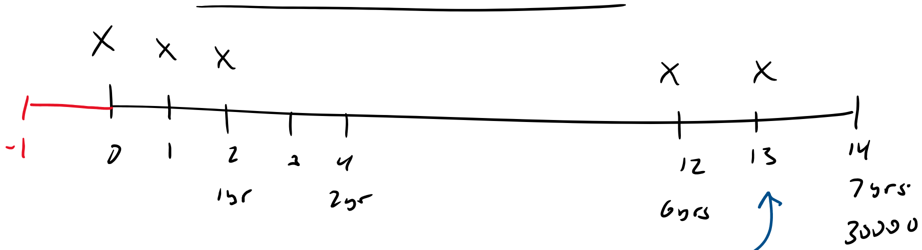
$$s_{\overline{n}|} = \ddot{s}_{\overline{n}|}v$$

Example: Sam wishes to accumulate \$30,000 in an account in 7 years. He will make deposits semiannually with the first deposit at time 0 and the last deposit at time 6.5. How large should the deposit be if the account earns a nominal rate of 8% compounded semianuualy.

$i^{(2)} = 8\%$

$\frac{i^{(2)}}{2} = 4\%$

$d = \frac{i}{1+i} = \frac{.04}{1.04}$




annuity Due.

$$30000 = X\ \ddot{s}_{\overline{14}|.04} = X \frac{(1+i)^{14}-1}{d}$$

$$X = \frac{30000\, d}{(1+i)^{14}-1} = \frac{30000\left(\frac{.04}{1.04}\right)}{(1.04)^{14}-1}$$

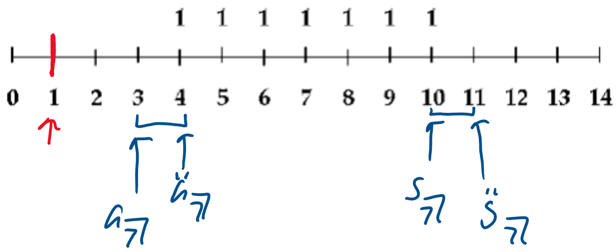
$X = \$1576.99$

annuity Imediate

$$30000 = X\, s_{\overline{14}|.04}\,(1+i)$$

$X = 1576.99$

Example: Suppose 7 payments of 1 are made at the end of the 4th through 10th periods, inclusive.




Find the value of the annuity

(a) at the end of the 1st period.

at $t=1$ $\quad a_{\overline{7}|} \cdot v^2 = \ddot{a}_{\overline{7}|} \cdot v^3 = s_{\overline{7}|} v^4 = \ddot{s}_{\overline{7}|} v^{10}$

Note: This in an example of a deferred annuity, since payments only commence after a deferred period.
Notation: ${}_{m|}a_{\overline{n}|}$ is n payments deffered after m periods.

at $t=1$

$a_{\overline{7}|} v^2 = {}_{2|}a_{\overline{7}|}$

at $t=1$

$\ddot{a}_{\overline{7}|} v^3 = {}_{3|}\ddot{a}_{\overline{7}|}$

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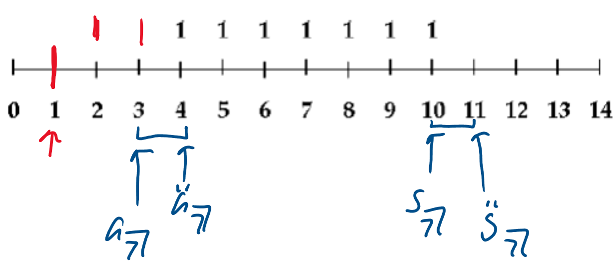


value at $t=1$ of all pmts

$a_{\overline{9}|}$

9 pmts at $t=1$ $\quad$ 2 pmts at $t=1$

$a_{\overline{9}|} - a_{\overline{2}|} = {}_{2|}a_{\overline{7}|} = a_{\overline{7}|} v^2$

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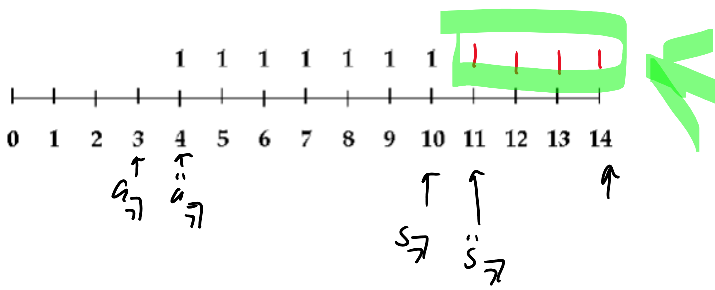
$\ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{3}|} = \ddot{a}_{\overline{7}|} v^3$

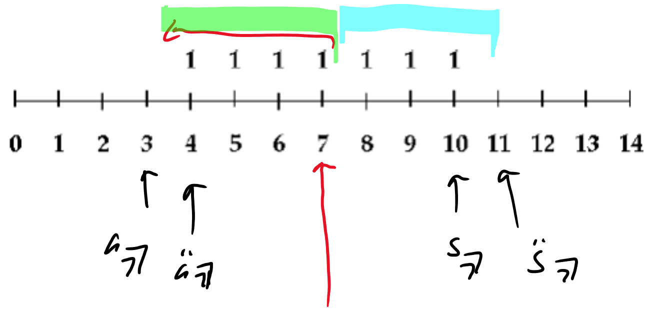


add pmts at $t = 11, 12, 13, 14$

(b) at the end of the 14th period.

$$s_{\overline{7}|}(1+i)^4 = \ddot{s}_{\overline{7}|}(1+i)^3 = s_{\overline{11}|} - s_{\overline{4}|}$$

$$= \ddot{s}_{\overline{10}|} - \ddot{s}_{\overline{3}|}$$




$s_{\overline{7}|} \cdot v^3$

$a_{\overline{7}|} \cdot (1+i)^4$

$\ddot{a}_{\overline{7}|} \cdot (1+i)^3$

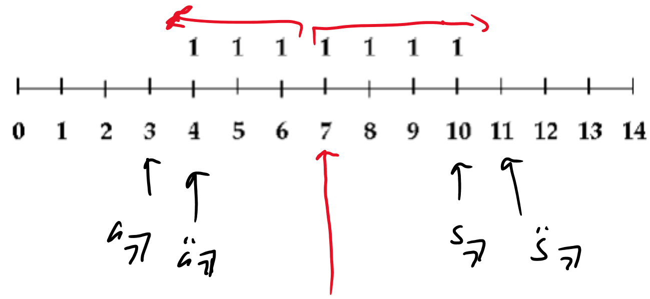
$\ddot{s}_{\overline{7}|} \cdot v^4$

(c) At the end of the 7th period.

Annuity immediate (pmt end of interval)

$$s_{\overline{4}|} + a_{\overline{3}|}$$

annuity due (pmt. at beginning)




$s_{\overline{7}|} \cdot v^3$

$a_{\overline{7}|} \cdot (1+i)^4$

$\ddot{a}_{\overline{7}|} \cdot (1+i)^3$

$\ddot{s}_{\overline{7}|} \cdot v^4$

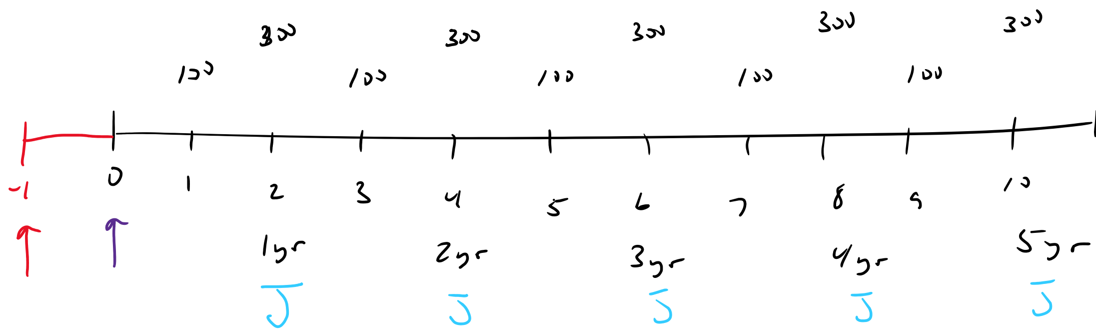
$$\ddot{s}_{\overline{3}|} + \ddot{a}_{\overline{4}|}$$

Semi annual pmts of 100 at $t = 1, 3, 5, 7, 9$

pmt of 300 at $t = 2, 4, 6, 8, 10$

$i^{(2)} = 6\%$

find value at $t = 0.$




$i^{(2)} = 6\%$

annual eff. $= 6.09\%$

$$PV = \underbrace{300\, a_{\overline{5}|6.09\%}}_{\text{time } 0} + \underbrace{100\, a_{\overline{5}|6.09\%}}_{\substack{\text{time } -1 \\ \text{ie } -1/2 \text{ yr.}}}(1 + .0609)^{1/2}$$

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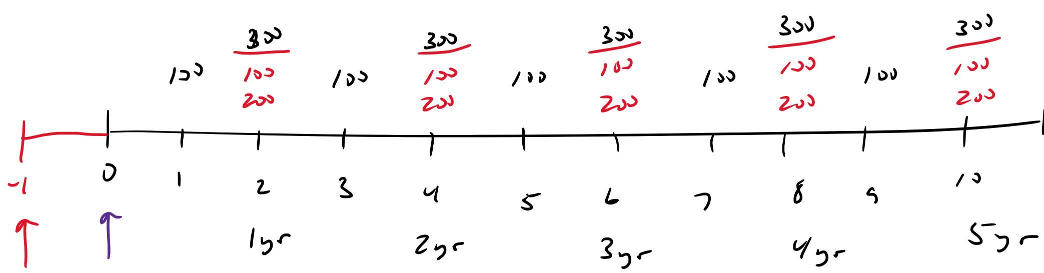
$$J = 100\left(1 + \frac{.06}{2}\right)^{1} + 300 \qquad PV = J\, a_{\overline{5}|6.09\%}$$

find value at $t = 0.$




$i^{(2)} = 6\%$

annual eff. $= 6.09\%$

$$\underbrace{100\, a_{\overline{10}|\frac{i^{(2)}}{2}}} + \underbrace{200\, a_{\overline{5}|i_{eff}}}$$