3.5: Perpetuities

A **perpetuity** is an annuity whose payment continue forever. The term of the annuity is not finite.

$v = \frac{1}{1+i}$

$v^n = \frac{1}{(1+i)^n}$

- The present value of a perpetuity-immediate is denoted by $\underbrace{a_{\overline{\infty}|}}$.

$a_{\overline{\infty}|i} = \frac{1}{i}$

$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|i} = \lim_{n\to\infty} \frac{1-v^n}{i} = \underbrace{\frac{1}{i}}$

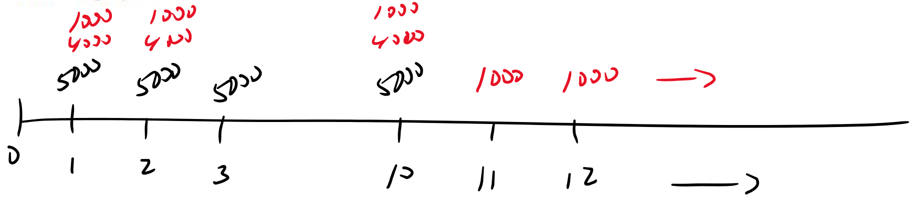
Invest at t=0 to generate pmts of 1 forever.

- The present value of a perpetuity-due is denoted by $\ddot{a}_{\overline{\infty}|}$.

$\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$

Note: Accumulated values do not exist since payments continue forever.

Example: Payments of $5000 are received at the end of each year for 10 years, after which payments of $1000 are received at the end of each year forever. The annual effective interest rate is 9%. Determine the present value of these payments.



$i_{eff} = 9\%$

$$\underbrace{5000\, a_{\overline{10|}.09}}_{\text{time zero}} + \underbrace{1000\, a_{\overline{\infty|}.09}}_{\text{valued at time } t=10}\, v^{10}$$

$$\underbrace{4000\, a_{\overline{10|}.09}}_{\text{Time 0}} + \underbrace{1000\, a_{\overline{\infty|}.09}}_{\text{at time 0.}}$$

$$5000\left[\frac{1-(1.09)^{-10}}{.09}\right] + \frac{1000}{.09}\cdot(1.09)^{-10}$$

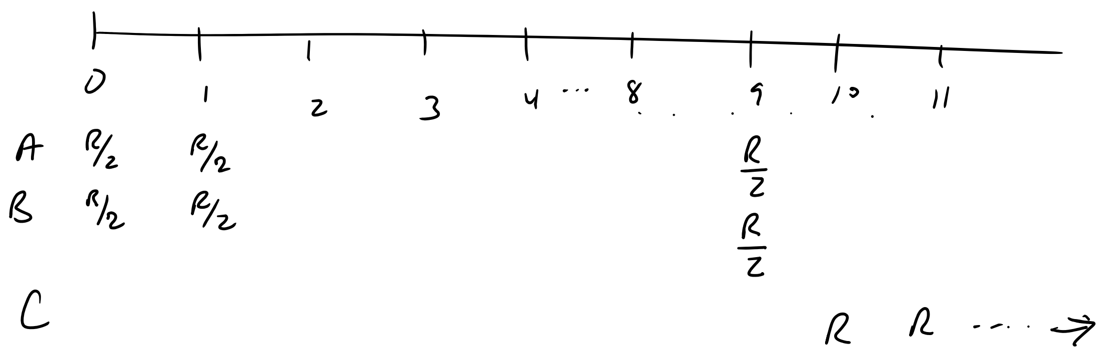
$$4000\left[\frac{1-(1.09)^{-10}}{.09}\right] + \frac{1000}{.09}$$

$$\boxed{\$36,781.74}$$

Example: Bob wishes to leave an inheritance to 3 charities in a series of level payments at the beginning of each year, continuing in perpetuity. He wants Charity A and B to share the payments equally for 10 years, and then all payments revert to charity C. If the shares received by all three charities have the same value at the time of bequest, find the annual effective interest rate i.

find
i annual eff

pmt = R



PV A)

$$PV = \frac{R}{2}\,\ddot{a}_{\overline{10|}i}$$

$$= \frac{R}{2}\left\{\frac{1-v^{10}}{d}\right]$$

PV C)

$$PV = R\,a_{\overline{\infty|}i}\cdot v^{9} = \frac{R}{i}v^{9}$$

$$PV = R\,\ddot{a}_{\overline{\infty|}i}\cdot v^{10} = \frac{R}{d}v^{10}$$

$$PV\,A = PV\,C$$

$$\frac{R}{2}\left\{\frac{1-v^{10}}{d}\right] = \frac{R}{d}v^{10}$$

$$\frac{1}{2}\left\{1-v^{10}\right] = v^{10}$$

$$\frac{1}{2} - \frac{1}{2}v^{10} = v^{10}$$

$$\frac{1}{2} = \frac{3}{2}v^{10}$$

$$\frac{2}{3}\cdot\frac{1}{2} = v^{10}$$

$$\frac{1}{3} = v^{10} = \frac{1}{(1+i)^{10}}$$

$$3 = (1+i)^{10}$$

$$i = 3^{1/10} - 1 = 11.612\%$$

3.6: Unknown Time

How do you interpret annuity symbols for non-integral terms?

Consider for n an integer and $0 < k < 1$.

$$a_{\overline{n+k}|} = \frac{1-v^{n+k}}{i} = \frac{1-v^n+v^n-v^{n+k}}{i} = \frac{1-v^n}{i} + \frac{v^n-v^{n+k}}{i}$$

$$a_{\overline{n+k}|} = a_{\overline{n}|} + \frac{v^{n+k}(v^{-k}-1)}{i}$$

$$a_{\overline{n+k}|} = a_{\overline{n}|} + \frac{(1+i)^k-1}{i}\, v^{n+k}$$

The above notation represent n payments of 1 and then a final payment at time $n+k$ of $\frac{(1+i)^k-1}{i}$.

In practice payments at non-integer periods are usually not done. Usually a balloon payment or a drop payment is made.

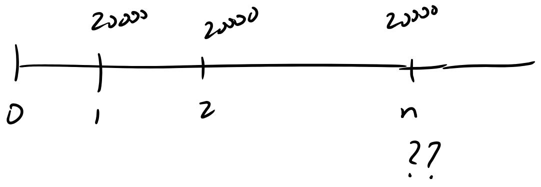
- A **balloon payment** is a payment that combines the regular payment and a smaller additional payment made at the same time as the last regular payment.
- A **drop payment** is a smaller additional payment that is made one period after the last regular payment.

annual eff. at 6%

Example: Bob has \$180,000 in an account and plans to withdraw \$20,000 at the end of each year for as long as possible. If $i = 6\%$, find how many regular payments can be made and find the amount of the smaller payment.

PV = 180,000

(a) to be paid in addition to the last regular payment. What is the balloon payment?



find n)

$$180000 = 20000\, a_{\overline{n}|6\%}$$

$$9 = \frac{1 - v^n}{.06}$$

$$v = \frac{1}{1.06}$$

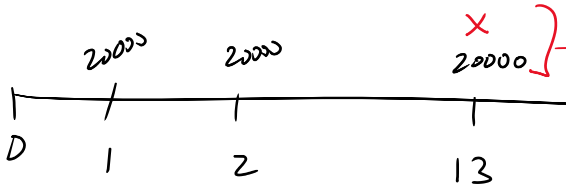
$$9(.06) = 1 - v^n$$

$$v^n = 1 - 9(.06)$$

$$n = \frac{\ln(1 - 9(.06))}{\ln(v)} = 13.32664$$

13 full pmts

K = 0.32664 when partial pmt happens after t=13



Balloon pmt is $\boxed{20000 + X}$

$$180000 = 20000\, a_{\overline{13}|6\%} + X \cdot v^{13}$$

$$\vdots$$

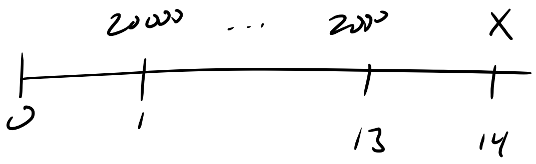
$$X = \$6,284.33$$

$$180000(1+i)^{13} = 20000\, s_{\overline{13}|} + X$$

Answer: 26,284.33

Example: Bob has \$180,000 in an account and plans to withdraw \$20,000 at the end of each year for as long as possible. If $i = 6\%$, find how many regular payments can be made and find the amount of the smaller payment.

(b) to be paid one year after the last regular payment. i.e. a drop payment.



$$180000 = 20000\, a_{\overline{13|}} + Xv^{14}$$

$$180000(1+i)^{14} = 20000\, s_{\overline{13|}}(1+i) + X$$

$$\underbrace{X = 6661.39}_{\text{drop payment}}$$

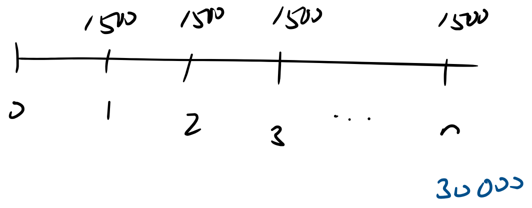
Example: Bob has \$180,000 in an account and plans to withdraw \$20,000 at the end of each year for as long as possible. If $i = 6\%$, find how many regular payments can be made and find the amount of the smaller payment.

(c) to be paid during the year following the last regular payment. i.e. paid during the next period consistent with the fractional time.

last pmt paid at $n = 13.32664$

$$20000 \, \frac{(1+i)^{.32664} - 1}{i} = \$6405.0885$$

Example: A fund of $30,000 is to be accumulated by making deposits of $1500 at the end of every year as long as necessary. If the fund earns an effective interest rate of 10%, find how many regular deposits will be necessary and the size of the final deposit made one year after the last regular deposit.



$$30000 = 1500\, s_{\overline{n}|10\%}$$

$$30000 = 1500\,\frac{(1.10)^n - 1}{.10}$$

$$A = \frac{30000(.10)}{1500} + 1 = 1.10^n$$

$$\frac{\ln(A)}{\ln(1.10)} = n = 11.526$$

TVM solver

N = solve. → n = 11.526

I = 10%

PV = 0

PMT = −1500

FV = 30000

P/Y = C/Y = 1

pmt: end

11 full pmts

---

what extra is needed at t=12?

time 11 $1500\, s_{\overline{11}|10\%} = \$27{,}796.75$

time 12 $27796.75(1.10) = \$30{,}576.43$

so no extra pmt needed

3.7: Unknown Rate of Interest

The best way to determine an unknown rate of interest for a basic annuity is to use a financial calculator.

$$\text{pmt}\, s_{\overline{n}|i} = \text{pmT}\, \frac{(1+i)^{n}-1}{i}$$

Note: There are two approximation formulas for this in the book. Don't worry about them.

Example: At what rate of interest convertible monthly is is $10,000 the present value of $300 paid at the beginning of every month for 6 years?

$i^{(12)}$

$$10000 = 300\, \ddot{a}_{\overline{72}|i}$$

i = monthly eff. Rate per period $\rightarrow \frac{i^{(12)}}{12}$

$$10000 = 300\, \frac{1-\left(\frac{1}{1+i}\right)^{72}}{\frac{i}{1+i}}$$

use TVM solve

N = 72

I% = solve $\longrightarrow$ 2.58909% = $\frac{i^{(12)}}{12}$

PV = −10000

pmT = 300

FV = 0

pmt : Beginning

$$\boxed{i^{(12)} = 31.069\%}$$

Example: Find the rate of interest at which (a) $s_{\overline{2}|} = 2.35$, (b) $\ddot{s}_{\overline{4}|} = 4.7$.

N = 2

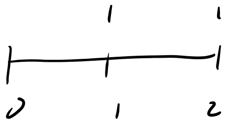
I = solve. → 35 %

PV = 0

pmt = -1

FV = 2,35

pmt : End.



$1(1+i) + 1 = 2.35$

$i = 0.35$

$i = 35\%$

b) 6.556 %

3.8: Varying Interest

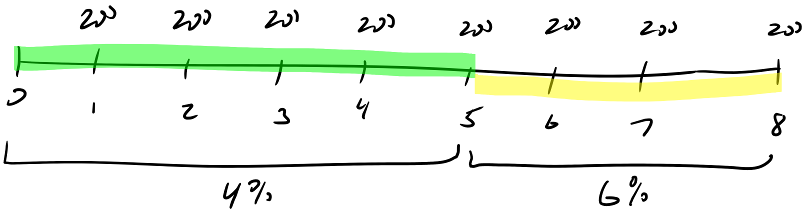
There are two common methods for varying interest rates for an annuity.

Portfolio Method: Assumes that the interest rate for any given period is the same for all payments whose value is affected by interest during that period.

Yield Curve Method: Assumes that each payment has an associated interest rate which remains level over the entire period for which present values(or accumulated values) are being computed.

Example: Find the accumulated value of an 8-year annuity-immediate of \$200 per year if the effective rate of interest is 4% for the first 5 years and 6% for the last three years.

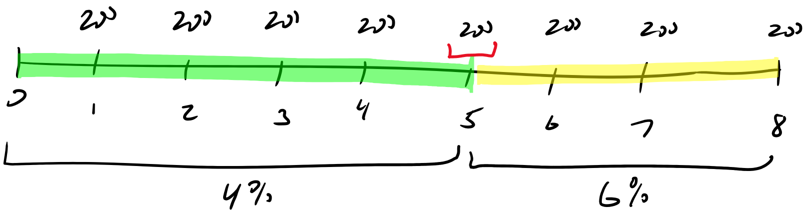
(a) Use the portfolio method.



Accumulated Amt. $200\,s_{\overline{5}|4\%}(1.06)^3 + 200\,s_{\overline{3}|6\%} = \$1926.91$

(b) Use the yield curve method.

assume deposit before Rate change



$$200(1.04)^7 + 200(1.04)^6 + 200(1.04)^5 + 200(1.04)^4 + 200(1.04)^3 + 200(1.06)^2$$

$$+ 200(1.06)^1 + 200$$

$$200\,s_{\overline{5}|4\%}(1.04)^3 + 200\,s_{\overline{3}|6\%}$$