Section 4.5: Continuous Annuities

An annuity where the frequency of payment becomes infinite is called a continuous annuity.

$\bar{a}_{\overline{n}|}$ is the present value of an annuity payable continuously for n interest conversion periods, such that the total amount paid during each interest conversion period is 1.

$\bar{a}_{\overline{n}|\,i}$

$i =$ annual eff. Rate of Interest

$0 \quad dt \quad n$

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{v^t}{\ln(v)}\bigg|_0^n = \frac{v^n - 1}{\ln(v)} = \frac{-(1-v^n)}{\ln(v)} = \frac{1-v^n}{-\ln\left(\frac{1}{1+i}\right)} = \frac{1-v^n}{\ln(1+i)} = \frac{1-v^n}{\delta} = \bar{a}_{\overline{n}|}$$

Alternate derivation: $\bar{a}_{\overline{n}|} = \lim_{m\to\infty} a^{(m)}_{\overline{n}|} = \lim_{m\to\infty} \frac{1-v^n}{i^{(m)}} = \frac{1-v^n}{\delta} = \frac{1-e^{-\delta n}}{\delta}$

$\delta = \ln(1+i)$

$e^{\delta} = 1+i$

$e^{-\delta} = v$

Notice: $\bar{\ddot{a}}_{\overline{n}|} = \lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|} = \lim_{m\to\infty} \frac{1-v^n}{d^{(m)}} = \frac{1-v^n}{\delta}$

$$\bar{\ddot{a}}_{\overline{n}|} = \bar{a}_{\overline{n}|}$$

$\bar{s}_{\overline{n}|}$ is the future value of an annuity payable continuously for n interest conversion periods, such that the total amount paid during each interest conversion period is 1.

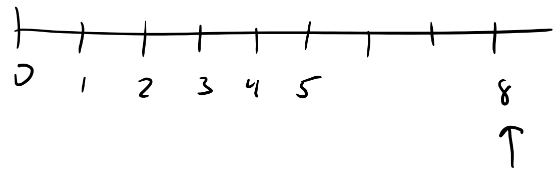
$$\bar{s}_{\overline{n}|} = \int_0^n (1+i)^t dt = \left.\frac{(1+i)^t}{\ln(1+i)}\right|_0^n = \frac{(1+i)^n}{\ln(1+i)} - \frac{1}{\ln(1+i)} = \frac{(1+i)^n - 1}{\ln(1+i)} = \frac{(1+i)^n - 1}{\delta} = \bar{s}_{\overline{n}|}$$

$$1+i = e^{\delta} \qquad \bar{s}_{\overline{n}|} = \frac{e^{\delta n} - 1}{\delta}$$

$(1+i) = e^{\delta}$

Example: Payments of $1000 per year are made continuously over a 5 year period. Find the accumulated value of the fund three years after the payments end, assuming interest is 8% per annum compounded continuously.

$\hookrightarrow \delta = .08$



$$1000\,\bar{s}_{\overline{5|}}\,(1+i)^3$$

$$= 1000\left(\frac{e^{.08(5)}-1}{.08}\right)\cdot e^{3(.08)} = 7815.40$$

Example: In 2004 and 2005 Susan deposits 12 every day into an account and in 2006 she deposits 15 every day in to the account. the account earns interest from the exact time of the deposit, with interest quoted as an effective rate of interest. The rates are 9% in 2004 and 2005 and 12% in 2006. Find the amount in the account, including interest, on December 31, 2006 using the approximation that deposits are made continuously.

2004 + 2005 annual deposits are $12(365) = 4380$

2006 annual deposits are $15(365) = 5475$

exact value

16502.59

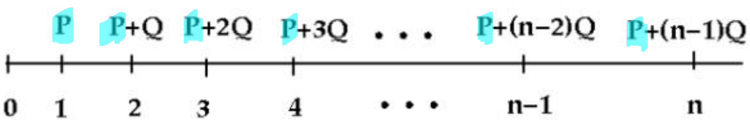
$$4380\ \bar{s}_{\overline{2}|\underbrace{.09}_{\text{annual eff}}} \cdot (1.12) + 5475\ \bar{s}_{\overline{1}|.12}$$

$$4380 \cdot \frac{(1.09)^2 - 1}{\ln(1.09)} \cdot (1.12) + 5475 \cdot \frac{(1.12)^1 - 1}{\ln(1.12)} = 16504.75$$

Now consider annuities that have varying payments.

Any type of varying annuity can be evaluated by taking the present value or the accumulated value of each payment separately and summing the results.

Consider a general annuity-immediate with a term of n periods in which payments begin at P and increase by Q per period thereafter. The interest rate is i per period.



present value let A = present value.

$$A = P a_{\overline{n}|i} + Qv^2 + 2Qv^3 + 3Qv^4 + \cdots + (n-2)Qv^{n-1} + (n-1)Qv^n$$

$$= P a_{\overline{n}|i} + Qv\left[\underbrace{v + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1}}_{T}\right]$$

$$T = v + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1}$$

$$-\left(vT = v^2 + 2v^3 + \cdots + (n-2)v^{n-1} + (n-1)v^n\right)$$

$$T - vT = v + v^2 + v^3 + \cdots + v^{n-1} - (n-1)v^n$$

$$T(1-v) = \underbrace{v + v^2 + v^3 + \cdots + v^{n-1} + v^n}_{} - nv^n$$

$$T(1-v) = a_{\overline{n}|} - nv^n$$

$$T = \frac{a_{\overline{n}|i} - nv^n}{1-v}$$

$$A = P a_{\overline{n}|i} + Qv \cdot \frac{a_{\overline{n}|} - nv^n}{1-v}$$

$$= P a_{\overline{n}|} + Q \frac{a_{\overline{n}|} - nv^n}{(1+i)(1-v)}$$

$(1+i)(1-v)$

$= 1+i - (1+i)v$

$= 1+i-1$

$= i$

present value $A = P a_{\overline{n}|} + Q \dfrac{a_{\overline{n}|} - nv^n}{i}$ Annuity Immediate

Annuity Imediate

$$\text{present value} = P a_{\overline{n}|} + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$$

$$\text{Future value} = \left(P a_{\overline{n}|} + Q\,\frac{a_{\overline{n}|} - nv^n}{i}\right)(1+i)^n$$

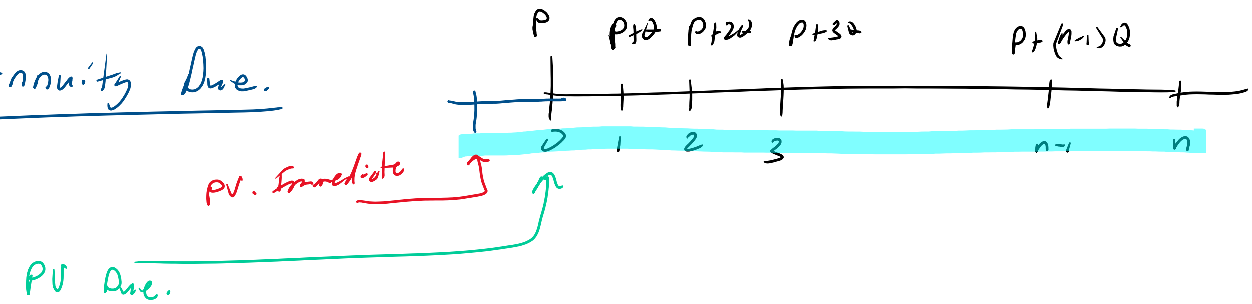
$P = 1^{st}$ pmt

Q = growth

$$= P a_{\overline{n}|}(1+i)^n + Q\,\frac{a_{\overline{n}|}(1+i)^n - nv^n(1+i)^n}{i}$$

$$\underline{FV} = P s_{\overline{n}|} + Q\left(\frac{s_{\overline{n}|} - n}{i}\right)$$

Annuity Due.



$$PV = \left[P\, a_{\overline{n}|} + Q\, \frac{a_{\overline{n}|} - n v^n}{i} \right] (1+i) \qquad d = \frac{i}{1+i}$$

$$= P\, a_{\overline{n}|} (1+i) + Q\, \frac{a_{\overline{n}|} - n v^n}{i} (1+i) \qquad \frac{1}{d} = \frac{1}{\frac{i}{1+i}} = \frac{1+i}{i}$$

$$PV = P\, \ddot{a}_{\overline{n}|} + Q\, \frac{a_{\overline{n}|} - n v^n}{d}$$

$$FV = \left[P\, \ddot{a}_{\overline{n}|} + Q\, \frac{a_{\overline{n}|} - n v^n}{d} \right] (1+i)^n \quad \text{or } FV = \left[P\, s_{\overline{n}|} + Q\, \frac{s_{\overline{n}|} - n}{i} \right] (1+i)$$

$$FV = P\, \ddot{s}_{\overline{n}|} + Q\, \frac{s_{\overline{n}|} - n}{d}$$

Page 4: increasing annuity

Two Special Cases: Annuity Immediate

- Increasing Annuity in which $P = 1$ and $Q = 1$

Present Value: $\underbrace{(Ia)_{\overline{n}|}}$

$$(Ia)_{\overline{n}|} = a_{\overline{n}|} + \frac{a_{\overline{n}|} - nv^n}{i} = \frac{i\,a_{\overline{n}|} + a_{\overline{n}|} - nv^n}{i} = \frac{a_{\overline{n}|}(1+i) - nv^n}{i}$$

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

Accumulated value: $(Is)_{\overline{n}|} = s_{\overline{n}|} + \dfrac{s_{\overline{n}|} - n}{i} = \dfrac{i\,s_{\overline{n}|} + s_{\overline{n}|} - n}{i}$

$$(Is)_{\overline{n}|} = \frac{s_{\overline{n}|}(1+i) - n}{i}$$

$$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

Page 5: decreasing annuity

$$a_{\overline{n|}} = \frac{1-v^n}{i}$$

$$i\,a_{\overline{n|}} = 1-v^n$$

- Decreasing Annuity in which $\underbrace{P = n \text{ and } Q = -1}$

Present Value: $(Da)_{\overline{n|}}$

$$(Da)_{\overline{n|}} = n\,a_{\overline{n|}} - \frac{a_{\overline{n|}} - nv^n}{i} = \frac{n\,i\,a_{\overline{n|}} - a_{\overline{n|}} + nv^n}{i}$$

$$= \frac{n(1-v^n) - a_{\overline{n|}} + nv^n}{i} = \frac{n - nv^n - a_{\overline{n|}} + nv^n}{i}$$

$$(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{i}$$

Accumulated value: $(Ds)_{\overline{n|}} = \left(\frac{n - a_{\overline{n|}}}{i}\right)(1+i)^n = \frac{n(1+i)^n - s_{\overline{n|}}}{i}$

Varying Perpetuities

Annuity Immediate

If $P > 0$ and $Q > 0$ the the present value of of an arithmetic varying perpetuity-immediate is

$$\text{PV} = \lim_{n\to\infty} Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{i} = \frac{P}{i} + \frac{Q}{i^2}$$

since $\lim_{n\to\infty} nv^n = 0$ if $|v| < 1$. can be shown by L'Hopitals.

Thus the notation/formula for an increasing perpetuity immediate is $(\text{Ia})_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2}$

Annuity Due

$$PV = P\ddot{a}_{\overline{n}|} + Q\frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$ for finite ~~the~~ payments

perpetuits

$$PV = \lim_{n\to\infty} P\ddot{a}_{\overline{n}|} + Q\frac{\ddot{a}_{\overline{n}|} - nv^n}{d} = \frac{P}{d} + \frac{Q}{id}$$

Note: All formulas are for annuity-immediate. Changing the i in the denominator of the above formulas to d will produce values for annuity-due.

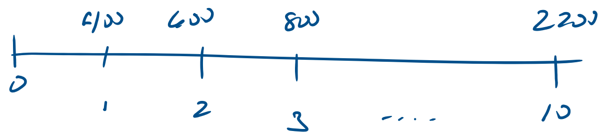
$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}(1+i) = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

$$(I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d}$$

$$(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d} \qquad (D\ddot{s})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}$$

Example: George receives \$400 at time 1, \$600 at time 2, \$800 at time 3 and so on until the final payment of \$2,200. Using an annual effective interest rate of 6%, determine the present value of these payments at time 0.

$i = 6\%$ $\qquad$ $n = 10$



$P = 400$

$Q = 200$

$400 + x\,200 = 2200$

$x\,200 = 1800$

$\underbrace{x = 9}_{\text{Increases}}$

total 10 pmts

$$PV = 400\, a_{\overline{10|}.06} + 200\, \frac{a_{\overline{10|}.06} - 10v^{10}}{.06}$$

$$= 1472.0174 + 7392.4816 = 8864.499$$

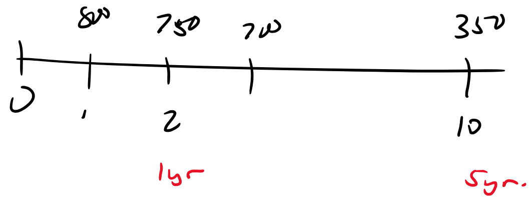
Example: George receives \$400 at time 1, \$600 at time 2, \$800 at time 3 and so on until the final payment of \$2,200. Using an annual effective interest rate of 6%, determine the present value of these payments at time 0.

shortcut where $P = Q$

$Q = 200$

$$PV = \underbrace{200\, a_{\overline{10|}}}_{\text{fixed part.}} + \underbrace{200\,(Ia)_{\overline{10|}}}_{\text{Increasing part.}} = 8864.499$$

Example: An annuity-immediate has semiannual payments of 800, 750, 700,..., 350 and $i^{(2)} = 0.16$. Find the present value of this annuity.



eff. semiannual Rate $= \frac{i^{(2)}}{2} = 8\%$

$P = 800$

$Q = -50$

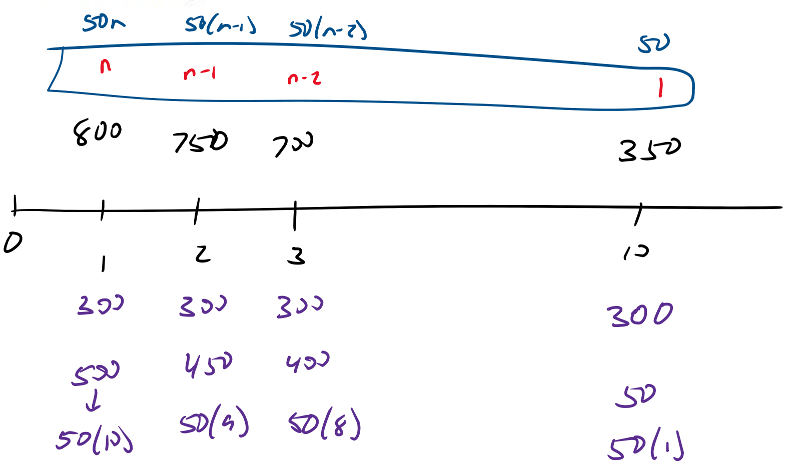
$$PV = 800\, a_{\overline{10|}.08} + (-50)\frac{a_{\overline{10|}.08} - 10v^{10}}{.08}$$

$$= 5368.064 - 1258.84$$

$$= 4069.22$$

Example: An annuity-immediate has semiannual payments of 800, 750, 700,..., 350 and $i^{(2)} = 0.16$. Find the present value of this annuity.

Shortcut.



$$\underbrace{300\, a_{\overline{10|}}}_{\text{fixed part.}} + \underbrace{50\,(Da)_{\overline{10|}}}_{\text{decreasing part.}} = 4069.22$$

Example: Find the present value of a perpetuity-immediate whose first payment is 100 and each subsequent payment increases by 25 if the rate per period is 5%.

$$PV = \frac{P}{i} + \frac{Q}{i^2} = \frac{100}{.05} + \frac{25}{(.05)^2} = 12000$$

Example: A perpetuity due has annual payments that start today with a payment of \$20 and increase by annual amounts of \$20 for ever. Assume an annual effective rate of 5%. Find the present value of this annuity.

Interest

$$PV = \frac{P}{d} + \frac{Q}{di}$$

$$d = \frac{i}{1+i}$$

$$= \frac{20}{\frac{.05}{1.05}} + \frac{20}{\frac{.05}{1.05}(.05)}$$

$$= 8820$$