Amortization Schedules and Sinking Funds

In this chapter, we discuss two methods of repaying a loan:

The Amortization Method. The borrower repays the lender by means of installment payments at periodic intervals. This process is called "amortization" of the loan.

The Sinking Fund Method. The borrower repays the lender by means of one lump-sum payment at the end of the term of the loan along with interest payments made in installments over the this period. It is also assumed that the borrower makes periodic payments into a fund, called a "sinking fund," which will accumulate to the amount of the loan to be repaid at the end of the term of the loan.

Section 5.2: Outstanding Loan Balance

If a loan is being repaid by the amortization method, the installment payments form an annuity whose present value is equal to the original amount of the loan.

$$\text{Loan} = L = Ra_{\overline{n}|}$$

We want to be able to determine the outstanding loan balance at any time.

Note: These terms are all equivalent: "outstanding loan balance", "outstanding principal," "unpaid balance," "remaining loan indebtedness."

Methods for Determining an Outstanding Loan Balance

Prospective Method: The outstanding loan balance at any point in time is equal to the present value at the date of the remaining installments payments.

Retrospective Method: The outstanding loan balance at any point in time is equal to the original amount of the loan accumulated to that date less the accumulated value at that date of all installment payments previously made.

$$L(1+i)^t - Rs_{\overline{t}|}$$

Note: Prospective and Retrospective methods provide the same outstanding ballance if the same assumptions about the amounts and timing of payments are made and the same rates of interest are used.

Notation:

B_t = outstanding loan balance at time t. (after t pmts)

B_t^p = outstanding loan balance according to the $\underline{\underline{\text{p}}}$rospective method.

B_t^r = outstanding loan balance according to the $\underline{\underline{\text{r}}}$etrospective method.

$L = R\, a_{\overline{n|}}$

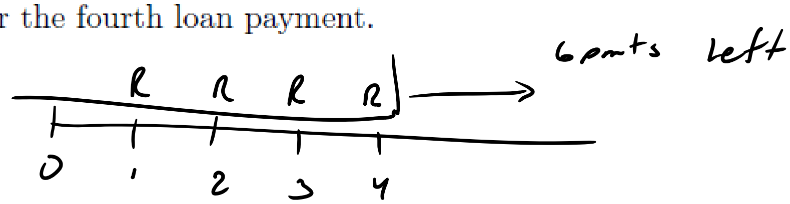
$$B_t^P = R\, a_{\overline{n-t|}}$$

$$B_t^R = L(1+i)^t - R\, s_{\overline{t|}}$$

$$= R\, a_{\overline{n|}} (1+i)^t - R\, s_{\overline{t|}}$$

Example: A bank makes a loan of $10,000. The loan will be paid back after 10 years with yearly payments of $1,358.68 at the end of each year. The annual effective interest rate is 6%. Calculate the amount of the loan left right after the fourth loan payment.

pmt 1358.68 = R



$$B_4^P = 1358.68\, a_{\overline{6}|6\%} = 6681.07$$

TVM solver

N = 6
I = 6%
PV = solve
pmt = -1358.68
FV = 0
pmt end

$$B_4^R = 10000(1+.06)^4 - 1358.68\, s_{\overline{4}|6\%} = 6681.07$$

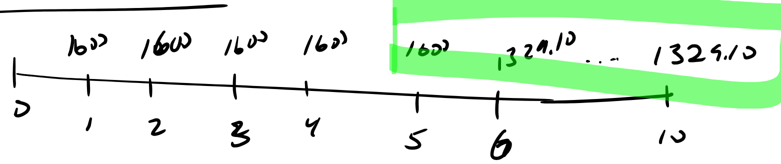
TVM solver

N = 4
I = 6%
PV = 10000
pmt = -1358.68
FV = solve
pmt end

Example: A \$10,000 loan is going to be paid back with 5 annual payments of \$1600 followed by 5 annual payments of \$1329.10. The annual effective interest rate if 8%.

A) Calculate the balance after 4 payments.

$i = 8\%$



$$B_4^P = 1600v + 1329.10\, a_{\overline{5}|}\, v$$

$$B_4^R = 10000(1.08)^4 - 1600\, s_{\overline{4}|8\%}$$

$\rightarrow$ 6395.11

B) Suppose right after the 4th payment, the borrower wishes to pay an extra \$1000 and then repay the balance over 6 years with revised level annual payments. Assuming the same effective rate of interest, find the revised annual payment.

still owe $= 6395.11 - 1000 = 5395.11$

$R =$ new payment

$$5395.11 = R\, a_{\overline{6}|8\%}$$

$$R = 1167.05$$

TVM

N = 6

I = 8%

PV = 5395.11

PMT = solve

FV = 0

pmt end

Section 5.3: Amortization schedules

If a loan is being repaid by the amortization method, each payment is partially repayment of principal and partially payment of interest.

An amortization schedule is a table which shows the division of each payment into principal and interest, together with the outstanding loan balance after each payment is made.

Example: A loan of $5000 is being repaid with level annual payments at the end of each year for 4 years at an annual effective rate of 6.09%. Make an amortization schedule for this loan.

$i = 6.09\%$ $\quad$ pmt $= R$ $\quad$ $L = R\,a_{\overline{4|}}$

$n = 4$ pmts. $\quad$ $5000 = R\,a_{\overline{4|}} \to R = 1445.93$

Period	Payment	Interest	Principal Repaid	Outstanding Balance
0	—	—	—	B_0 5000
1	R_1 1445.93	$I_1 = iB_0$ 304.50	$P_1 = R_1 - I_1$ 1141.43	$B_1 = B_0 - P_1$ 3858.57
2	R_2 1445.93	$I_2 = iB_1$ 234.99	$P_2 = R_2 - I_2$ 1210.94	$B_2 = B_1 - P_2$ 2647.63
3	R_3 1445.93	$I_3 = iB_2$ 161.24	$P_3 = R_3 - I_3$ 1284.69	$B_3 = B_2 - P_3$ 1362.94
4	R_4 1445.93	$I_4 = iB_3$ 83.00	$P_4 = R_4 - I_4$ 1362.93	B_4 .01
Totals	5783.72	783.73	4999.99	

Generalized Amortization Schedule

Suppose a loan of L is to be repaid with periodic payments of R_k, $1 \le k \le n$ and i is an effective rate per period. Then the following amortization schedule can be constructed:

Period	Payment	Interest Paid	Principal Repaid	Outstanding Loan Balance
0	–	–	–	$B_0 = L$
1	R_1	$I_1 = i \cdot B_0$	$P_1 = R_1 - I_1$	$B_1 = B_0 - P_1$
2	R_2	$I_2 = i \cdot B_1$	$P_2 = R_2 - I_2$	$B_2 = B_1 - P_2$
3	R_3	$I_3 = i \cdot B_2$	$P_3 = R_3 - I_3$	$B_3 = B_2 - P_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	R_t	$I_t = i \cdot B_{t-1}$	$P_t = R_t - I_t$	$B_t = B_{t-1} - P_t$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	R_{n-1}	$I_{n-1} = i \cdot B_{n-2}$	$P_{n-1} = R_{n-1} - I_{n-1}$	$B_{n-1} = B_{n-2} - P_{n-1}$
n	R_n	$I_n = i \cdot B_{n-1}$	$P_n = R_n - I_n$	$B_n = B_{n-1} - P_n$

$$B_3 = B_2 - P_3$$
$$= B_2 - (R_3 - I_3)$$
$$= B_2 - R_3 + i B_2$$
$$= B_2 + i B_2 - R_3$$
$$B_3 = B_2(1+i) - R_3$$

$$B_4 = B_3(1+i) - R_4$$
$$= \left(B_2(1+i) - R_3\right)(1+i) - R_4$$
$$B_4 = B_2(1+i)^2 - R_3(1+i) - R_4$$
$$= \underbrace{B_2(1+i)^2}_{\text{Balance at } t=2 \text{ accumulated 2 periods}} - \underbrace{\left[R_3(1+i) + R_4\right]}_{\text{accumulation of the 2 payments.}}$$

Now suppose a loan of L is to be repaid with n level payments of R and i is the effective rate per payment period.

$L = Ra_{\overline{n}|}$

$\frac{L}{a_{\overline{n}|}} = R$

Period	Payment	Interest Paid	Principal Repaid	Outstanding Loan Balance
0	–	–	–	$B_0 = L = Ra_{\overline{n}\|}$
1	R	$I_1 = iB_0$ $I_1 = R(1 - v^n)$	$P_1 = R - I_1$ $P_1 = Rv^n$	$B_1 = Ra_{\overline{n-1}\|}$
2	R	$I_2 = R(1 - v^{n-1})$	$P_2 = Rv^{n-1}$	$B_2 = Ra_{\overline{n-2}\|}$
3	R	$I_3 = R(1 - v^{n-2})$	$P_3 = Rv^{n-2}$	$B_3 = Ra_{\overline{n-3}\|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	R	$I_t = R(1 - v^{n-t+1})$	$P_t = Rv^{n-t+1}$	$B_t = Ra_{\overline{n-t}\|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	R	$I_{n-2} = R(1 - v^3)$	$P_{n-2} = Rv^3$	$B_{n-2} = Ra_{\overline{2}\|}$
$n-1$	R	$I_{n-1} = R(1 - v^2)$	$P_{n-1} = Rv^2$	$B_{n-1} = Ra_{\overline{1}\|}$
n	R	$I_n = R(1 - v)$	$P_n = Rv$	$B_n = 0$
TOTALS:	$\underbrace{nR}_{\text{Total Repaid}}$	$\underbrace{nR - Ra_{\overline{n}\|}}_{\text{Total Interest}}$	$Ra_{\overline{n}\|} = L$	

$\uparrow$ have a geometric factor of $(1+i)$

$I_1 = iB_0 = iRa_{\overline{n}|}$

$I_1 = R(1-v^n)$

$P_1 = Rv^n$

$P_2 = Rv^n(1+i)$

$= Rv^{n-1}$

$P_3 = P_2(1+i) = Rv^{n-1}(1+i)$

$P_3 = Rv^{n-2}$

P_1

$P_2 = P_1(1+i)$

$P_3 = P_2(1+i) = P_1(1+i)^2$

$P_4 = P_3(1+i) = P_1(1+i)^3$

$t_1 < t_2$

$P_{t_2} = P_{t_1}(1+i)^{t_2 - t_1}$

i.e. $P_5 = P_1(1+i)^4$

$P_1 + P_2 + P_3 + \cdots\cdots + P_n = L = Ra_{\overline{n}|}$

$P_1 + P_1(1+i) + P_1(1+i)^2 + \cdots + P_1(1+i)^{n-1} = L = Ra_{\overline{n}|}$

$P_1 s_{\overline{n}|} = L = Ra_{\overline{n}|}$

P_1, P_2, P_3

$n-3$ <u>terms</u>

$$B_3 = P_4 + P_5 + P_6 + \cdots + P_n$$

$$= P_4 + P_4(1+i) + P_4(1+i)^2 + \cdots + P_4(1+i)^{n-4}$$

$$B_3 = P_4 \, s_{\overline{n-3}|}$$

$$B_{t-1} = P_t \, s_{\overline{n-t+1}|}$$ (general form)

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Example: A loan is being repaid with payments of \$1,500 at the end of each quarter. If the loan will be repaid in 5 years and is being charged a nominal rate of 12% compounded quarterly, find principal and interest in the first half of the third year.

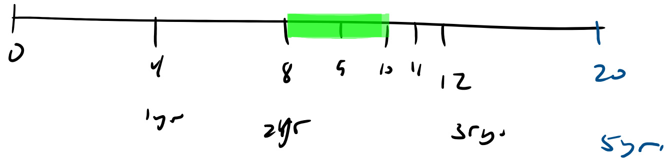
$L = ??$ $\quad L = R\,a_{\overline{20|}3\%}$

$R = 1500$

$n = 20$

$i^{(4)} = 12\%$

$\frac{i^{(4)}}{4} = \frac{12}{4} = 3\%$



Compute $P_9 + P_{10}$ and $I_9 + I_{10}$

find B_8 then we have what we need.

$$B_8 = R\,a_{\overline{12|}} = 1500\,\frac{(1-v^{12})}{.03} = \$14{,}931.01$$

$$I_9 = i\,B_8 = (.03)(14931.01) = 447.930179$$

$$P_9 = R - I_9 = 1500 - 447.93 = 1052.0658 = 1052.07$$

$$B_9 = B_8 - P_9 = 13878.936 = 13878.94$$

$$I_{10} = i\,B_9 = 416.37$$

$$P_{10} = R - I_{10} = 1083.63$$

$$P_9 + P_{10} = 2135.7072$$

$$I_9 + I_{10} = 864.298$$

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20 points total

$$P_9 + P_{10} = B_8 - B_{10} = 1500\,a_{\overline{12|}} - 1500\,a_{\overline{10|}}$$

$$= 14931.01 - 12795.304$$

$$= 2135.706$$

$$I_9 + I_{10} = 2(R) - (P_9 + P_{10})$$

$$= 2(1500) - 2135.706 = 864.298$$

Example: A loan is amortized by level payments made at the end of every year where the last payment will be a smaller final payment. The interest paid during the year 2015 was \$103.00 and the interest paid during the year 2016 ws \$98.00. The annual effective rate of interest for the loan is $i = 8\%$. Find the amount of the level loan payments.

Example: A loan is being repaid with a series of payments at the end of each quarter for 5 years. If the amount of principal in the third payment is \$100, find the amount of principal in the last 5 payments. The interest rate is $i^{(4)} = 10\%$.