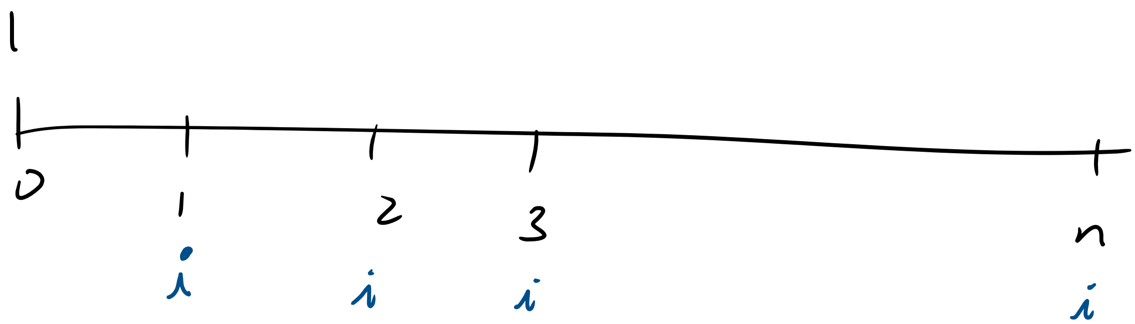
Section 7.4: Reinvestment Rates

Friday, March 27, 2020 8:29 PM

Up until now, we have always assumed that the lender can reinvest payments received from the borrower at a reinvestment rate equal to the original investment rate.

Now lets consider the situation in which the reinvestment rate is different from the rate at which the payments are made.

Consider the investment of 1 for n periods at rate i such that the interest is reinvested at rate j. Find the accumulated value at the end of n periods.



$$FV = 1 + i\, s_{\overline{n}|j}$$

$$= 1 + i\left(\frac{(1+j)^n - 1}{j}\right)$$

Now consider an investment of 1 at the end of each period where interest is paid at a rate of i but then reinvested at a rate of j.

$P = i$

$Q = i$

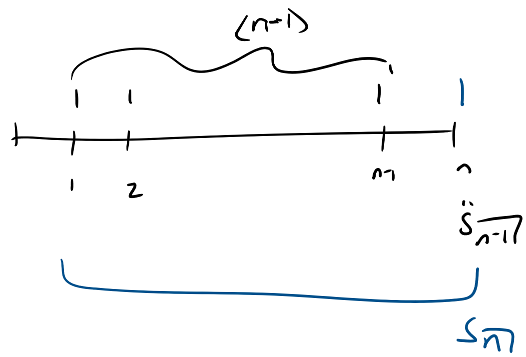
$i(Is)_{\overline{n-1}|}$

$$FV = 1(n) + i(Is)_{\overline{n-1}|j}$$

$$= n + i\,\frac{\ddot{s}_{\overline{n-1}|j} - (n-1)}{j}$$

$$= n + i\,\frac{\ddot{s}_{\overline{n-1}|j} + 1 - n}{j}$$

$$FV = n + i\,\frac{s_{\overline{n}|j} - n}{j}$$



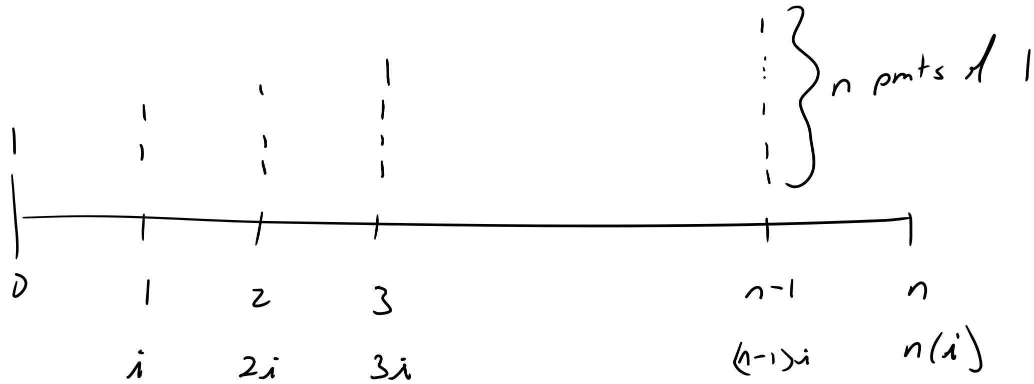
$P = i$

$Q = i$

$$\boxed{FV = n + i\,s_{\overline{n-1}|j} + i\left(\frac{s_{\overline{n-1}|} - (n-1)}{j}\right)}$$

$$\boxed{FV = P\,s_{\overline{n}|} + Q\,\frac{s_{\overline{n}|} - n}{i}}$$

Note: for an annuity-due with payments of 1 at the beginning of each period for n periods at a rate i with reinvestment of interest at rate j,



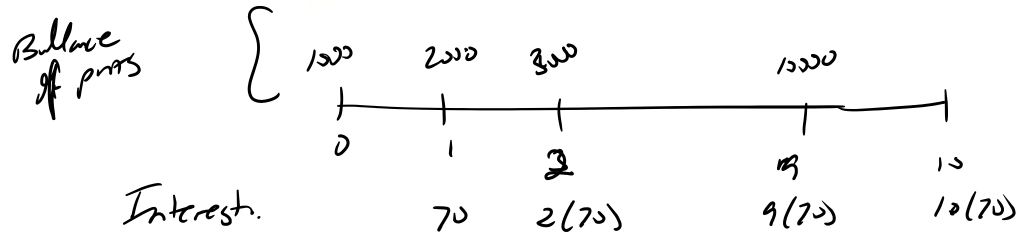
$$FV = n + i(Is)_{\overline{n}|j}$$

$$FV = n + i\, s_{\overline{n}|j} + i\left(\frac{s_{\overline{n}|j} - n}{j}\right) \rightarrow n + i\left(\frac{s_{\overline{n+1}|j} - (n+1)}{j}\right)$$

Example: Payments of \$1000 are invested at the beginning of each year for 10 years. The payments earn interest at 7% effective and the interest can be reinvested at 5% effective.

$1000(.07) = 70$

(a) Find the amount in the fund at the end of 10 years.
(b) Find the purchase price an investor should pay to produce a yield rate of 8% effective.



A)

$$FV = 1000(10) + 70\, s_{\overline{10}|5\%} + 70\left(\frac{s_{\overline{10}|5\%} - 10}{.05}\right)$$

$$FV = 14{,}489.50$$

$$\text{price}\,(1+.08)^{10} = 14489.50$$

$$\text{price} = 14489.50\,(1.08)^{-10}$$

$$= \$6711.44$$

Analysis for loans

From chapter 5, we know the loan amount, L, is the present value of a series of n payments, R, at interest rate i.

$L = Ra_{\overline{n}|i}$

From the viewpoint of the lender:

The lender invested the original loan amount L and is receiving n payments of R. Suppose the lender is only able to reinvest these payments at rate j.

What is the adjusted yield rate, i', which reflects the impact of the different reinvestment rate?

$$L(1+i')^n = Rs_{\overline{n}|j}$$

Example: A loan of \$10,000 is being repaid with 25 level annual payments with interest charged at 8% per year effective. find the yield to the lender if they are only able to reinvest the payments received at 4% per year.

Loan 10,000

$n = 25$

$r = 8\%$

need pmt.

$10000 = R\, a_{\overline{25|}\, 8\%}$

$R = \dfrac{10000}{a_{\overline{25|}\, 8\%}} = 936.79$

TVM

N = 25

I = 8%

PV = 10000

PMT = solve

FV = 0

$10000(1+i')^{25} = 936.79\, s_{\overline{25|}\, 4\%}$

$i' = 5.59627\%$ annual eff

TVM

N = 25

I = 4%

PV = 0

PMT = 936.79

FV = $\boxed{}$ = 39013.17

TVM

N = 25

I = solve

PV = −10000

PMT = 0

FV = 39013.17

Example: A loan of $1,000 is being repaid with interest only payments at the end of each year and the principal is repaid at the end of 10 years. The effective rate of interest for the loan is 9% per annum. Find the yield to the lender if they are only able to reinvest the payments received at 7% per year.

$$pmt = 1000\,(9\%)$$
$$= 90 \qquad \text{Interest only.}$$

$$1000\,(1+i')^{10} = \boxed{90\, s_{\overline{10}|7\%} + 1000}$$

$$i' = 8.4157\% \quad \text{annual eff.}$$

TVM

$N = 10$

$I =$ solve

$PV = -1000$

$Pmt = 0$

$FV = \boxed{}$

$n = 120$

$\boxed{\text{monthly}}$ eff.-Rate

Analysis for bonds

Suppose The coupons from a bond are reinvested at rate j. To find the yield rate on the bond considering reinvestment, i', we solve.

$$\text{Price}\,(1+i')^{n} = Fr\, s_{\overline{n}|j} + C$$

Example: A 20 year bond with 8% semiannual coupons and a face amount of 100 is quoted at a purchase price of 70.400. Assume the coupons can only be reinvested at 7% convertible semiannually. find the yield rate taking into account reinvestment rates.

price $= 70.400$

$F = 100$

$r = 4\%$

$C = 100$

coupon $= Fr = 4$

$n = 20(2) = 40$

$j = 3.5\%$

$$70.4\,(1+i')^{40} = 4\, s_{\overline{40}|3.5\%} + 100$$

$$i' = 4.6773\% = \frac{i^{(2)}}{2}$$

$$\boxed{i^{(2)} = 9.3546\%}$$

annual eff $= 9.5733\%$

Section 7.5: Interest Measurement of a Fund

A common requirement in practical work is the determination of the yield rate earned by an investment fund.

To use the basic definition of effective rate of interest, we assume the principal remains constant throughout the period, and all interest earned is paid at the end of the period.

Goal: Find the effective rate of interest earned over one measurement period.

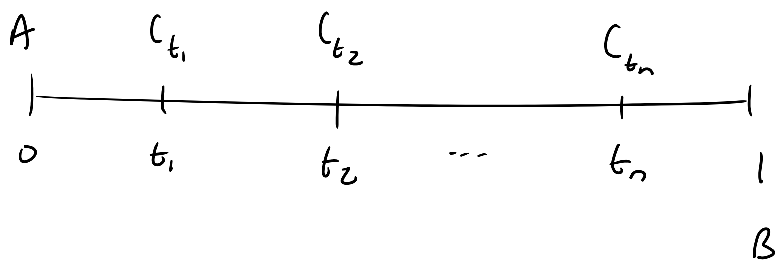
A = the amount in the fund at the beginning of the period.

B = the amount in the fund at the end of the period.

I = the amount of interest earned during the period.

C_t = the net amount of principal contributed at time t (positive or negative), where $0 \le t \le 1$

$C = \sum_t C_t$ the total net amount of principal contributed during the period (positive or negative)



$i =$ yield Rate (IRR)

$$B = A + C + I \longrightarrow I = B - A - C$$

$$I = Ai + C_{t_1}\left[(1+i)^{1-t_1} - 1\right] + \cdots + C_{t_n}\left[(1+i)^{1-t_n} - 1\right]$$

$$I = Ai + \sum_t C_t\left[(1+i)^{1-t} - 1\right]$$

Can use the eq to find the actual yield Rate (IRR) for the fund.

If approximate answers are sufficient, we can simplify the formula for the amount of interest earned by assuming the contributions earn simple interest instead of compound interest.

$$I = Ai + \sum_t C_t\left[(1+i)^t - 1\right]$$

$$I \approx Ai + \sum_t C_t(1-t)i = i\left(A + \sum_t C_t(1-t)\right)$$

$$i \approx \frac{I}{A + \sum_t C_t(1-t)}$$

This formula will not produce a true effective rate of interest because of the simple interest assumption. As long as the C_t's are small in comparison to A, it will be a good approximation.

The quantity $A + \sum_t C_t(1-t)$ is referred to as the **exposure to risk**. Can be interpreted as the average amount of principal invested in the period.

The approximation formula for the IRR using simple interest is also known as the **dollar-weighted rate of return**.

$$IRR = i \approx i^{DW} = \frac{I}{A + \sum_t C_t(1-t)}$$

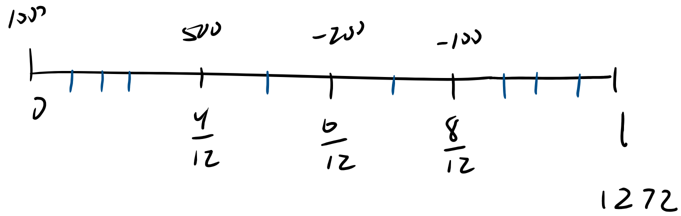
Example: At the beginning of the year an investment fund was established with an initial deposit of \$1000. A new deposit of \$500 was made at the end of four months. Withdrawals of \$200 and \$100 were made at the end of six months and eight months, respectively. The amount in the fund at the end of the year is \$1272. Find the approximate effective rate of interest (dollar-weighted rate of return) earned by the fund during the year.

$A = 1000$

$B = 1272$

$C = 500 - 200 - 100$

$C = 200$



$B = A + C + I$

$I = B - A - C$

$= 1272 - 1000 - 200$

$= 72$

$$i^{dw} = \frac{I}{A + \sum C_t(1-t)}$$

$$= \frac{72}{1000 + 500\left(\frac{8}{12}\right) + (-200)\left(\frac{6}{12}\right) + (-100)\left(\frac{4}{12}\right)}$$

$$i^{dw} = \frac{72}{1200} = .06 \to 6\%$$

To get Actual IRR

use cash flows

$CF_0 = 1000$
$C01 = 0$
$F01 = 3$
$C02 = 500$
$F02 = 1$

$C03 = 0$
$F03 = 1$
$C04 = -200$
$F04 = 1$
$C05 = 0$
$F05 = 1$

$C06 = -100$
$F06 = 1$
$C07 = 0$
$F07 = 3$
$C08 = -1272$
$F08 = 1$

IRR cpt

$\hookrightarrow .48720837\% = \frac{i^{(12)}}{12}$

$\hookrightarrow i^{(12)} = 5.8465\%$

$\hookrightarrow$ eff $= 6.005738\%$

This formula, due to the summation in the denomination, can be laborious to calculate for large number of principal contributions during the period.

$$i^{DW} = \frac{I}{A + \underbrace{\sum_t C_t(1-t)}}$$

A further simplification is to assume the total net amount of principal contributed during the period, C, is done at $t = 1/2$.

$A + C + I = B$

$C = B - A - I$

$$i^{DW} \approx \frac{I}{A + C\frac{1}{2}} = \frac{2I}{2A + C} = \frac{2I}{A + B - I}$$

approx with i^{DW}

Example: Find the effective rate of interest earned during a calendar year by an insurance company with the following data:

Assets, beginning of the year	\$10,000,000	⟷ starting Ballance
Premium income	+ 1,000,000	
Net investment income	510,000	← Interest
Policy benefits	− 420,000	
other expenses	− 180,000	

$A = 10,000,000$

$I = 510\,000$

$C = 1000000 - 420000 - 180000$

$B = 10000000 + 1000000 - 420000 - 180000 = 10,510,000$

$$i^{DW} = \frac{2I}{A+B-I} = \frac{2(510000)}{10000000 + 10510000 - 510000} = .05 \to 5\%$$