Section 7.6: Time-weighted rates of interest

Consider an account that is worth \$1000 at the start, worth \$500 at the end of 6 months, and worth \$1000 at the end of the year. If no principal is deposited or withdrawn during the year, then the yield rate for the entire year is zero.

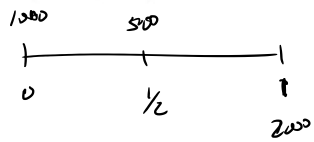
$$1+i^{tw} = \left(\frac{500}{1000}\right)\left(\frac{2000}{500+500}\right)$$

$$1+i^{tw} = \frac{1}{2} \cdot 2 = 1$$

$$i^{tw} = 1-1 = 0 \rightarrow 0\%$$

Suppose an additional principal of \$500 is added at the 6 month period into this account. Computing the yield rate gives.

$1000(1+i) + 500(1+i)^{1/2} = 2000$ solving for i gives $i = 40.69\%$



CF0 = 1000

C01 = 500

F01 = 1

C02 = −2000

F02 = 1

IRR cpt

↳ $18.614\% = \frac{i^{(2)}}{2}$

Semi annual eff rate

$i^{(2)} = (18.614\%)2$

annual eff. Rate = 40.69%

$A = 1000$

$B = 2000$

$C = 500$

$I = 2000 - 1000 - 500 = 500$

$$i^{DW} = \boxed{\frac{I}{A + \frac{1}{2}C}} = \frac{500}{1000 + \frac{1}{2}(500)} = 40\%$$

Suppose an additional principal of $250 is removed at the 6 month period into this account. Computing the yield rate gives.

$1000(1+i) - 250(1+i)^{1/2} = 500$ solving for i gives $i = -28.92\%$

CF0 = 1000
C01 = -250
F01 = 1
C02 = -500
F02 = 1

$A = 1000$
$C = -250$
$B = 500$
$I = 500 - 1000 - -250$
$= -250$

$$i^{(DW)} = \frac{-250}{1000 + \frac{1}{2}(-250)} = -28.75\%$$

These methods for computing the yield rates are sometimes called **dollar-weighted rates of interest** since the amounts invested and the timing of the investments affects the rates.

$$1 + i^{TW} = \frac{500}{1000}\left(\frac{500}{500 - 250}\right) = \frac{1}{2}\left(\frac{500}{250}\right) = \frac{1}{2}(2)$$

$1 + i^{TW} = 1$

$i^{TW} = 0\%$

An investment fund manager generally does not have control over the timing or amounts of cash inflows and outflows for a fund.

A **time-weighted rate of interest**, i^{TW}, is often used to compare the relative performance of various investment fund managers since this method eliminates the impact of money flows in and out of the fund.

Example: Consider an investment of \$1 at the beginning of a year. Suppose during the year is divided into three time periods with i_k being the effective rate of the indicated time period: $i_1 = 4\%$, $i_2 = 3\%$, and $i_3 = 5\%$. Compute the yield for the year.

$$1(1+i_1)(1+i_2)(1+i_3) =$$

$$1(1.04)(1.03)(1.05) = 1(1+i^{TW})$$

$$1.12476 = 1 + i^{tw}$$

$$i^{Tw} = .12476 \longrightarrow \underline{i^{TW} = 12.476\%}$$

Consider the investment fund with $m-1$ principal withdrawals/deposits made at times during the year. Let C_k be the net contributions to the fund at time t_k where $k = 1, 2, ..., m-1$. Let B_k be the fund balance just before each contribution with B_0 being the fund balance at the start and B_m is the fund balance at the end.

	$t_0 = 0$	t_1	t_2	t_{m-1}	$t_m = 1$
Contribution		C_1	C_2	C_{m-1}	
Fund Value	B_0	B_1	B_2	B_{m-1}	B_m
(1+ **yield rate**)	$\frac{B_1}{B_0}$	$\frac{B_2}{B_1+C_1}$			

$$B_0(1+i_1) = B_1$$

$$\boxed{1+i_1 = \frac{B_1}{B_0}}$$

$$i_1 = \frac{B_1}{B_0} - 1$$

$$i_1 = \frac{B_1 - B_0}{B_0}$$

$$(B_1 + C_1)(1+i_2) = B_2$$

$$\boxed{(1+i_2) = \frac{B_2}{B_1+C_1}}$$

$$\boxed{1+i^{TW} = \left(\frac{B_1}{B_0}\right)\cdot\left(\frac{B_2}{B_1+C_1}\right)\cdot\left(\frac{B_3}{B_2+C_2}\right)\cdot \;\ldots\; \left(\frac{B_m}{B_{m-1}+C_{m-1}}\right)}$$

Example: A pension fund receives contributions and pays benefits from time to time. The fund value is reported after every transaction and at the end of the year. The details for 2009 are as follows.

	Date	Amount
Fund Values:	Januart 1	1,000,000
	March 1	1,240,000
	September 1	1,600,000
	November 1	1,080,000
	Jan 1, 2010	900,000
Contribution Received	February 28	200,000
	August 31	200,000
Benifits Paid	October 31	500,000
	December 31	200,000

Find the time-weighted rate of return.

	1/1	2/28, 3/1	8/31, 9/1	10/31, 12/1	12/31, 1/1/10
Fund Ballance after C	100	124	160	108	90
Contribution C		+20	+20	-50	-20
Ballance Before C	100	104	140	158	110

$$(1+i^{TW}) = \left(\frac{104}{100}\right)\left(\frac{140}{104+20}\right)\left(\frac{158}{140+20}\right)\left(\frac{110}{158-50}\right)$$

$$= 1.18098865$$

$$i^{TW} = 18.0988\%$$

If each month is 1/12

$i^{DW} = 17.39\%$

Section 7.7: Portfolio Methods and Investment Year Methods

Suppose that an investment fund pools money from several individuals or corporations and makes investments on behalf of them. i.e. a pension fund.

The fund faces the question: How to allocate the returns between different identities? There are two main ways to allocate interest to the various accounts: the **portfolio method** and the **investment year method**.

For the portfolio method an average rate based on the earnings of the entire fund is computed and credited to each account. This does not depend on when the money was put into the account.

This method may not be favorable for periods with rising interest rates. The portfolio may contain low-yielding investments which would give the fund a low average yield. Thus new investors are less likely to invest in the fund.

Let i^y denote the annual interest rate credited in year y. If x is invested at the beginning of year y then the balance at the beginning of year $y+t$ is

$$x(1+i^y)(1+i^{y+1})(1+i^{y+2})\cdots(1+i^{y+t-1})$$

Example: Suppose that an investment account credits investors using the portfolio method with the annual rates in the following table.

Calendar Year	Portfolio rates
y	i^y
2000	4.50%
2001	5.50%
2002	4.00%
2003	6.50%

Suppose that 100 was invested on January 1, 2000. Find the balance on of the account on January 1, 2002. on July 1, 2002.

1/1/02 $\quad 100(1+.045)(1+.055) = 110.2475$

7/1/02 $\quad 100(1.045)(1.055)(1.04)^{1/2} = 112.4308$

The investment year method, also called the new money method, is where the fund keeps track of both the year of the investment and the annual interest rates earned by that investment. This method produces a two dimensional table of interest rates. Due to the potential size of the table for long-term investments, this method is usually truncated after a fixed number of years and then is converted to the portfolio method.

Calendar year of original investment	Investment year rates %				Portfolio rates	Calendar year of portfolio rates
y	i_1^y	i_2^y	i_3^y	i_4^y	i^{y+4}	y+4
2000	4.25	4.35	4.47	4.57	4.70	2004
2001	4.56	4.73	4.75	4.98	4.04	2005
2002	4.05	4.04	4.13	4.17	4.24	2006
2003	4.45	4.15	4.23	4.36	4.44	2007
2004	4.25	4.35	4.55	5.25	5.15	2008
2005	4.35	4.70	5.75	5.30		
2006	5.15	6.10	5.80			
2007	6.25	5.15				
2008	5.35					

Suppose an investment of $1000 is made on January 1, 2001.

Calendar year of original investment	Investment year rates %				Portfolio rates	Calendar year of portfolio rates
y	i_1^y	i_2^{y+1}	i_3^{y+2}	i_4^{y+3}	i^{y+4}	y+4
2000	4.25	4.35	4.47	4.57	4.70	2004
2001	4.56	4.73	4.75	4.98	4.04	2005
2002	4.05	4.04	4.13	4.17	4.24	2006
2003	4.45	4.15	4.23	4.36	4.44	2007
2004	4.25	4.35	4.55	5.25	5.15	2008
2005	4.35	4.70	5.75	5.30		
2006	5.15	6.10	5.80			
2007	6.25	5.15				
2008	5.35					

Suppose an investment of $1000 is made on January 1, 2001.

(a) Find the balance on January 1, 2003.

$$1000(1.0456)(1.0473) = 1095.999$$

Calendar year of original investment	Investment year rates %				Portfolio rates	Calendar year of portfolio rates
y	i_1^y	i_2^y	i_3^y	i_4^y	i^{y+4}	y+4
2000	4.25	4.35	4.47	4.57	4.70	2004
2001	4.56	4.73	4.75	4.98	4.04	2005
2002	4.05	4.04	4.13	4.17	4.24	2006
2003	4.45	4.15	4.23	4.36	4.44	2007
2004	4.25	4.35	4.55	5.25	5.15	2008
2005	4.35	4.70	5.75	5.30		
2006	5.15	6.10	5.80			
2007	6.25	5.15				
2008	5.35					

Suppose an investment of \$1000 is made on January 1, 2001.

(b)Find the balance on January 1, 2007.

2001 2002 2003 2004 2015 2006

$1000(1.0456)(1.0473)(1.0475)(1.0498)(1.0404)(1.0424)$

$= 1305.966$

Calendar year of original investment	Investment year rates %				Portfolio rates	Calendar year of portfolio rates
y	i_1^y	i_2^y	i_3^y	i_4^y	i^{y+4}	y+4
2000	4.25	4.35	4.47	4.57	4.70	2004
2001	4.56	4.73	4.75	4.98	$\boxed{4.04}$	2005
2002	4.05	4.04	4.13	$\boxed{4.17}$	4.24	2006
2003	4.45	4.15	$\boxed{4.23}$	4.36	4.44	2007
2004	4.25	$\boxed{4.35}$	4.55	5.25	5.15	2008
2005	$\boxed{4.35}$	4.70	5.75	5.30		
2006	5.15	6.10	5.80			
2007	6.25	5.15				
2008	5.35					

Suppose an investment of $1000 is made on January 1, 2001.

(c) What interest rates are credited in the year 2005?

2005: 4.35%

2004: 4.35%

2003: 4.23%

2002: 4.17%

2001 + Before: 4.04%