Appendix J.2: The Dot Product

Definition: The **dot product** of two nonzero vectors **a** and **b** is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between the vectors **a** and **b**, $0 \le \theta \le \pi$. If either **a** or **b** is **0**, then we define $\mathbf{a} \cdot \mathbf{b} = 0$.

Example: Find $\mathbf{a} \cdot \mathbf{b}$ if $|\mathbf{a}| = 4$, $|\mathbf{b}| = 10$, and $\theta = \dfrac{\pi}{6}$

Example: Explaine what the dot product tells us if $\mathbf{a} \cdot \mathbf{b}$ is positive? is negative?

Definition: The **work** done by a force, **F**, in moving an object from point P to point Q, or with displacement $\mathbf{D} = \overrightarrow{PQ}$, is given by $\mathbf{W} = \mathbf{F} \cdot \mathbf{D}$.

Example: Find the work of using a force of 10N to move a block 3 meters if the force is applied at an angle of 25^o to the ground. (Assume that the ground is level.)

Definition: Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular (orthogonal) if and only if $\mathbf{a} \cdot \mathbf{b} = 0$

Example: Are the vectors $\mathbf{a} = \langle 3, 7 \rangle$ and $\mathbf{b} = \langle -5, 2 \rangle$ orthogonal?

Definition: Alternate definition for the dot product of two vectors. For two vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ then $\mathbf{a} \cdot \mathbf{b} = a_1 * b_1 + a_2 * b_2$

Proof of alternate definition:(optional). Let $(a) = \langle a_1, a_2 \rangle$ and $(b) = \langle b_1, b_2 \rangle$

$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta$

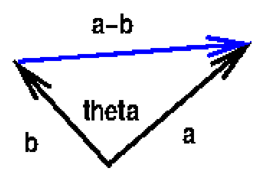
$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b}$

$(a_1 - b_1)^2 + (a_2 - b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2\mathbf{a} \cdot \mathbf{b}$

$2\mathbf{a} \cdot \mathbf{b} = a_1^2 + a_2^2 + b_1^2 + b_2^2 - (a_1^2 - 2a_1b_1 + b_1^2) - (a_2^2 - 2a_2b_2 + b_2^2)$

$2\mathbf{a} \cdot \mathbf{b} = 2a_1b_1 + 2a_2b_2$

$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$



Properties of the Dot Product: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and m is a scalar, then

$\mathbf{a} \cdot \mathbf{a} = \lvert\mathbf{a}\rvert^2$	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	$\mathbf{0} \cdot \mathbf{a} = 0$
$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$	$(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$	

Definition: The **orthogonal compliment** of $\mathbf{a} = \langle a_1, a_2 \rangle$ is $\mathbf{a}^{\perp} = \langle -a_2, a_1 \rangle$

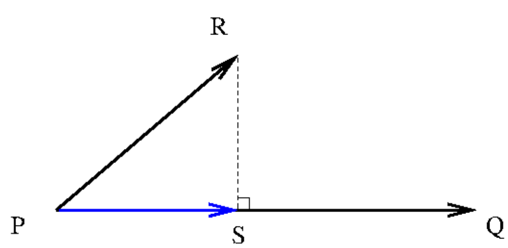
Example: What value(s) of x will make $\langle x, 4 \rangle$ and $\langle x, 7x \rangle$ orthogonal?

Example: A constant force $\mathbf{F} = 2\mathbf{i} + 4\mathbf{j}$, in Newtons, is used to move an object from $A(2,5)$ to $B(7,9)$. Find the work done if the distance between the points is measured in meters.

Example: Find the angle between $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$.

Scalar Projection and Vector Projection

The vector projection of $\mathbf{b} = \overrightarrow{PR}$ onto $\mathbf{a} = \overrightarrow{PQ}$, denoted as $\text{proj}_{\mathbf{a}}\mathbf{b}$, is the vector $\overrightarrow{PS}$.



Example: Find the scalar projection and the vector projection of $\mathbf{b} = \langle 3, 2 \rangle$ onto $\mathbf{a} = \langle 4, 6 \rangle$

Example: Find a vector $\mathbf{m}$ such that $\text{comp}_{\mathbf{n}^{\perp}} \mathbf{m} = 2$ and $\mathbf{n} = \langle 5, 12 \rangle$

Example: Find the vector projection of $\mathbf{b} = \langle 3, 2 \rangle$ onto $\mathbf{i}$

Example: Find a unit vector in the same direction as the projection of $\mathbf{b} = \langle 5, 1 \rangle$ onto $\mathbf{a} = \langle -1, 1 \rangle$

Example: Using vectors, find the distance from the point $(1, 0)$ to the line $y = 2x + 4$