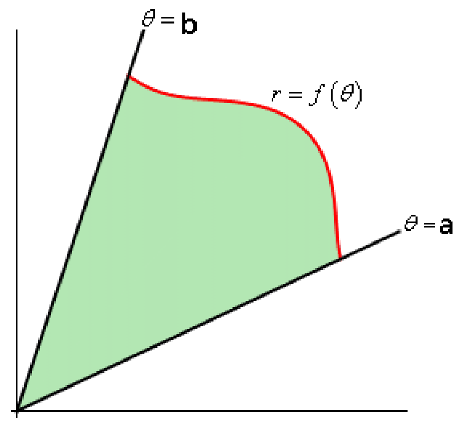
Section 10.4: Areas and Length in Polar Coordinates

We would like to find the area of the region that is between the pole (origin) and the polar equation $r = f(\theta)$ from $\theta = a$ to $\theta = b$.



To be able to find this area we start back with the area of a circle being $A = \pi r^2$.

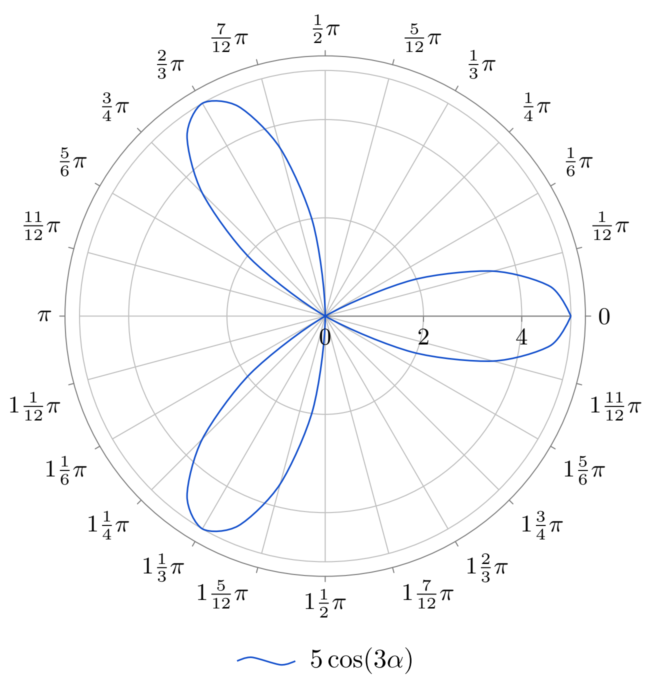
A sector of a circle, which is a part of the circle formed by the central angle θ, has an area that is proportional to the whole circle.

$A = \left(\frac{\theta}{2\pi}\right) \pi r^2 = \frac{1}{2} r^2 \theta$

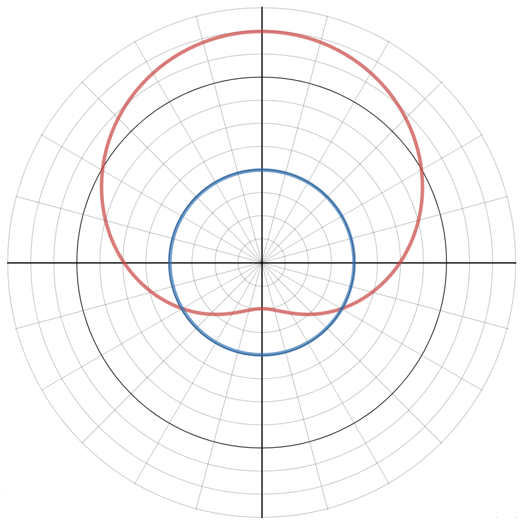
Now partition the region(on the right) where $\theta_1 = a$ to $\theta_n = b$. The area of each of the smaller sectors is given by $A_i = \frac{1}{2} r_i^2 \Delta\theta$. Then area of the region is approximated by $A \approx \sum \frac{1}{2} r_i^2 \Delta\theta$.

Thus the area of the region is $A = \displaystyle\int_a^b \frac{1}{2} r^2 d\theta$, where $r = f(\theta)$.

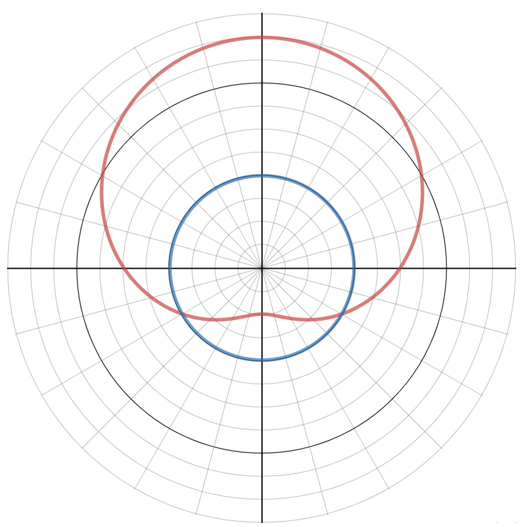
Example: Find the area of one petal of the graph $r = 5\cos(3\theta)$.



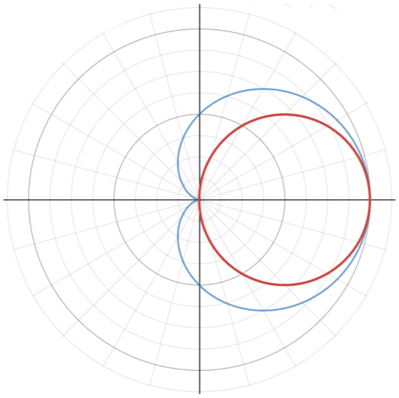
Example: Find the area inside $r = 3 + 2\sin\theta$ and outside the circle $r = 2$.



Example: Find the area inside the circle $r = 2$ and outside $r = 3 + 2\sin\theta$



Example: Setup the integral(s) that give the area above the x-axis and inside $r = 2 + 2\cos\theta$ and outside $r = 4\cos\theta$



Arc Length

From section 10.2 we know the length of a curve is $L = \int_a^b ds$ where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$.

Find the arc length of the polar curve $r =$ for $a \le \theta \le b$. Once again we assume that the curve is traced exactly once.

We start with $x = r\cos\theta$ and $y = r\sin\theta$ or $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$
We know the formula for ds.

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta = \cdots \text{lots of algebra} \cdots = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Example: Find the length of the curve $r = \theta$ for $0 \leq \theta \leq 1$.

Example: Find the length of the curve $r = -4\sin\theta$ for $0 \leq \theta \leq \frac{2\pi}{3}$

Example: Setup the integral that would give the length of the curve that forms one of the loops for $r = \sin(2\theta)$.