Section 11.1: Sequences

Definition: A **sequence** is a list of numbers written in a definite order.
$a_1, a_2, a_3, ...$ or $\{a_n\}_{n=1}^{\infty}$

Example: Find a general formula for these sequences.

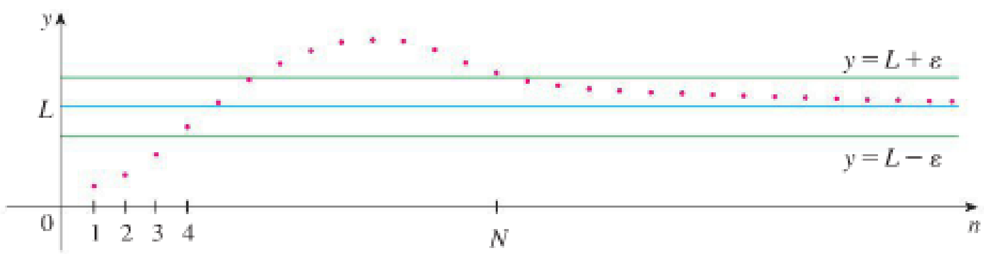
A) $\left\{\frac{5}{9}, \frac{6}{16}, \frac{7}{25}, \frac{8}{36}, ...\right\}$

B) $\left\{\frac{3}{4}, \frac{6}{11}, \frac{9}{18}, \frac{12}{25}, \frac{15}{32}, ...\right\}$

C) $\{1, 1, 2, 3, 5, 8, 13, 21, ...\}$

Definition: A sequence $\{a_n\}$ is said to have the limit L, written $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$, if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say that the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Definition: If $\{a_n\}$ is a sequence, then $\lim_{n\to\infty} a_n = L$ means that for every $\epsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.



Example: Do these sequences converge or diverge.

A) $\{(-1)^n\}_{n=1}^{\infty}$

B) $\{\cos(2n\pi)\}$

C) $\left\{\dfrac{3n}{n+2}\right\}_{n=5}^{\infty}$

THEOREM If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Example: Does the sequences converge or diverge? If it converges, give the value.

A) $\left\{ \dfrac{n^2}{\ln(3+e^n)} \right\}$

B) $\left\{ \dfrac{3n}{n+2} + \dfrac{n^2}{n^2+1} \right\}$

Limit Laws for Convergent Sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n * \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad if \lim_{n\to\infty} b_n \neq 0$$

Squeeze Theorem for Sequences: If $a_n \leq b_n \leq c_n$ for $n \geq n_o$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Example: Does the sequence a_n converge or diverge? If it converges, give the value.

A) $a_n = \dfrac{(-1)^n n^2}{n^2+1}$

B) $a_n = \dfrac{(-1)^n 3n}{n^2+5}$

C) $a_n = \dfrac{n!}{n^n}$

Example: Find the values of r so that $\{r^n\}$ converges. Determine what the series will converge to.

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is $a_1 < a_2 < a_3 < ....$ It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonoic** if it is either increasing or decreasing.

Example: Show that the sequence $a_n = \dfrac{n}{n^2+4}$ is a decreasing sequence.

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$. It is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$. If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

Question: If a sequence is convergent, is the sequence bounded?

Example: You are given that the sequence given by $a_1 = \sqrt{5},\ \ a_{n+1} = \sqrt{5 + a_n}$ is increasing and bounded above by 4.

Find $\lim_{n\to\infty} a_n$

Example: You are told that the sequence given by $a_1 = 1,\ \ a_{n+1} = 3a_n - 1$ is increasing. Does this sequence converge?

Example: Assume that this sequence will converge. Give the exact value that it will converge to.

$a_1 = -1$ $$a_{n+1} = \frac{1}{5}\left(a_n + \frac{44}{a_n}\right)$$