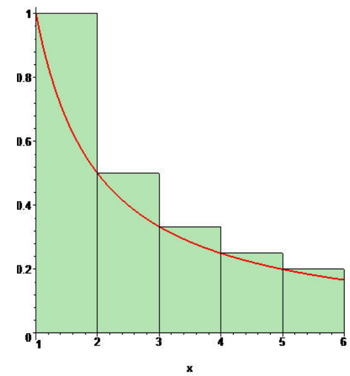
Section 11.3: The Integral Test and Estimates of Sums

Note: In this section all series have positive terms.

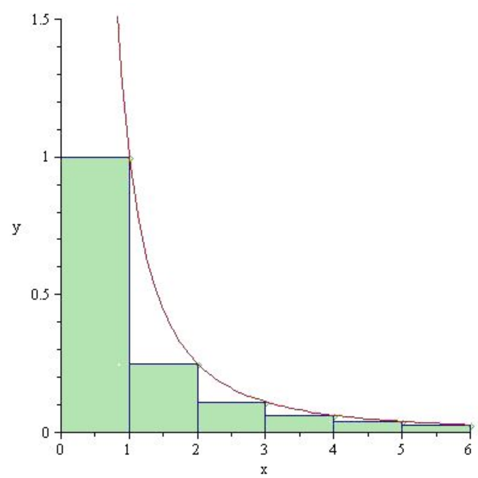
This graph was used to discuss the Harmonic series in section 11.2. We saw that

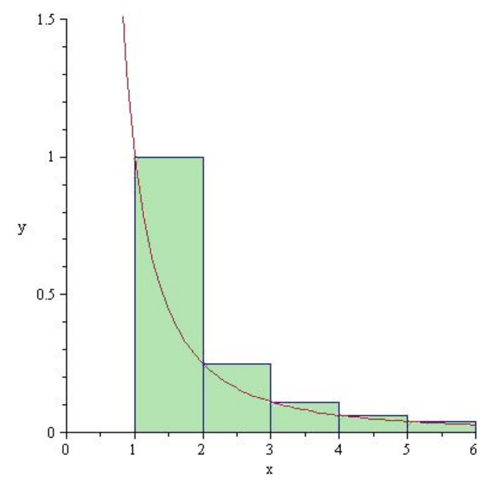
$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_{1}^{\infty} \frac{1}{x}\, dx > 0$$

and concluded that the Harmonic series diverged.



Here are the graphs for the function $f(x) = \dfrac{1}{x^2}$. Does the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converge or diverge?





The Integral Test Suppose f is **continuous, positive, decreasing function on** $[1, \infty)$, or on $[A, \infty)$, and let $a_n = f(n)$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int\limits_1^{\infty} f(x)\, dx$ is convergent. In other words:

(a) If $\int\limits_1^{\infty} f(x)\, dx$ is convergent then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int\limits_1^{\infty} f(x)\, dx$ is divergent then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Example: Assume that $\sum\limits_{n=1}^{\infty} a_n$ is a series where $f(n) = a_n$ and $\int\limits_1^{\infty} f(x)\, dx$ converges to the number L.

Does this mean that the series $\sum\limits_{n=1}^{\infty} a_n$ converge to L?

Example: Do these series converge or diverge?

A) $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{1+n^2}$

B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

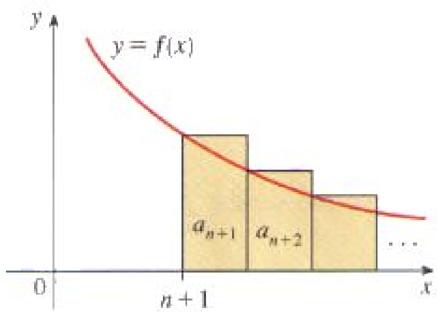
C) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n}$

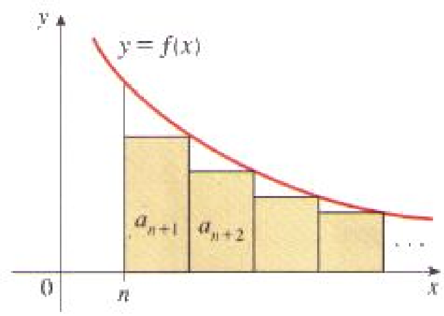
Remainder Estimate For The Integral Test Suppose that $\sum a_n$ converges by the Integral Test to the number s. If s_n is a partial sum that approximates this series, define R_n to be the remainder, i.e. $s_n + R_n = s$, then we get the following bounds on the remainder and the sum:

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n + a_{n+1} + a_{n+2} + ... = s$$

$$\int_{n+1}^{\infty} f(x)\ dx \leq\ R_n\ \leq \int_{n}^{\infty} f(x)\ dx$$

$$s_n + \int_{n+1}^{\infty} f(x)\ dx \leq\ s\ \leq s_n + \int_{n}^{\infty} f(x)\ dx$$





Example: Given that this series converges by the Integral test. $\displaystyle\sum_{i=1}^{\infty} \frac{30}{i^4}$.

A) Find the bounds on R_{10}

B) What bounds are on the sum for this series?

C) If you wanted the error to be less than 0.005, what is the smallest value of n should you use?

Example: The following series converges by integral test. Determine the smallest number of terms that should be used if the maximum error of the partial sum should be less than 0.05?

$$\sum_{i=1}^{\infty} \frac{1}{i^3}.$$