Section 11.6: Absolute Convergence and the Ratio and Root Tests

Definition: A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Definition: A series $\sum a_n$ is called **conditionally convergent** if the series $\sum |a_n|$ is divergent and the series $\sum a_n$ is convergent.

Theorem: If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2+1}$$

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

The Ratio Test:

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, with $0 \leq L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** (and therefore convergent).

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: If the limit for the ratio test is 1, then this test fails to give any information. Try something else.

Consider the results of the ratio test for two of our known $p-$series.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$, where $a_n = \frac{1}{n^2}$, converges since $p > 1$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)^2}{1/n^2}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \text{ by L'Hopitals.}$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n}$, where $a_n = \frac{1}{n}$, diverges since $p \leq 1$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)}{1/n}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1 \text{ by L'Hopitals.}$$

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$$

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} n!e^{-n}$$

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(n+1)^3}{(2n)!}$$

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(-4)^n n!}{7 * 12 * 17 * ... * (5n+2)}$$

Example: The series $\sum a_n$ is defined recursively by

$a_1 = 1 \qquad a_{n+1} = \dfrac{(2+\cos(n))a_n}{\sqrt{n}}$ for $n \geq 1$.

Is the series is absolutely convergent, conditionally convergent, or divergent?

The Root Test: (not covered in this course but is in the textbook)

(a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, with $0 \le L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** (and therefore convergent).

(b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: If the limit for the root test is 1, then this test fails to give any information. Try something else.

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n+7}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+3}{3n+7}\right)^n} = \lim_{n\to\infty} \left(\frac{4n+3}{3n+7}\right) = \frac{4}{3} > 1$$

By the root test the series will diverge.