Section 7.4: Additional Problems Solutions

1. First factor the denominator.

$$\frac{x^2-4x+4}{(x^2+2x)(x^2-25)^2(x^2+9)} = \frac{x^2-4x+4}{x(x+2)(x+5)^2(x-5)^2(x^2+9)}$$

Answer: $\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+5} + \frac{D}{(x+5)^2} + \frac{E}{x-5} + \frac{F}{(x-5)^2} + \frac{Gx+H}{x^2+9}$

2. Since the degree of the denominator is not larger than the degree of the numerator, the first step is long division.

$$\begin{array}{r|l} & \qquad\qquad\qquad\qquad 1 \\ \hline x^4+2x^2+1 & x^4+2x^3+0x^2+0x+7 \\ & -(x^4+0x^3+2x^2+0x+1) \\ \hline & \quad 2x^3-2x^2 \qquad\quad +6 \end{array}$$

$$\frac{x^4+2x^3+7}{x^4+2x^2+1} = 1 + \frac{2x^3-2x^2+6}{x^4+2x^2+1}$$

Now factor the denominator and then do do the decomposition.

$$1 + \frac{2x^3-2x^2+6}{x^4+2x^2+1} = 1 + \frac{2x^3-2x^2+6}{(x^2+1)^2}$$

Answer: $1 + \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

3. Since the degree of the denominator is larger than the degree of the numerator, long division is not needed.

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\frac{1}{x^2(x+1)} = \frac{Ax(x+1)}{xx(x+1)} + \frac{B(x+1)}{x^2(x+1)} + \frac{Cx^2}{(x+1)x^2}$$

$$1 = Ax(x+1) + B(x+1) + Cx^2 \qquad \text{equation 1}$$

$$1 = A(x^2+x) + B(x+1) + Cx^2 \qquad \text{equation 2}$$

Method of equating coefficents: use equation 2 to get the followingequations.

x^2 |	$0 = A + C$
x |	$0 = A + B$
constant |	$1 = B$

Solving these equations gives $A = -1$, $B = 1$, and $C = 1$.

Evaluation Method: use equation 1 to get the following.

evaluate $x = 0$ to get $1 = B$

evaluate $x = -1$ to get $1 = C$

To solve for A either plug in another value for x or use the equations in the Equate Coefficents Method to find that $A = -1$.

$$\int \frac{1}{x^3+x^2}\,dx = \int \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)\,dx = -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

4. The first step is doing the long division.

$$\begin{array}{r|llll} & & & & 1 \\ \hline x^3+4x^2 & x^3+7x^2 & +2x & +8 & \\ & -(x^3+4x^2) & \downarrow & \downarrow & \\ \hline & 3x^2 & +2x & +8 & \end{array}$$

$$\frac{x^3+7x^2+2x+8}{x^3+4x^2} = 1 + \frac{3x^2+2x+8}{x^3+4x^2}$$

Thus $\displaystyle\int \frac{x^3+7x^2+2x+8}{x^3+4x^2}\,dx = \int \left(1+\frac{3x^2+2x+8}{x^3+4x^2}\right)dx = \int\left(1+\frac{3x^2+2x+8}{x^2(x+4)}\right)dx$

Now do the partial fraction decomposition of the fraction $\dfrac{3x^2+2x+8}{x^2(x+4)}$

$$\frac{3x^2+2x+8}{x^2(x+4)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+4}$$

$$\frac{3x^2+2x+8}{x^2(x+4)} = \frac{Ax(x+4)}{xx(x+4)}+\frac{B(x+4)}{x^2(x+4)}+\frac{Cx^2}{(x+4)x^2}$$

$3x^2+2x+8 = Ax(x+4)+B(x+4)+Cx^2$ equation 1

$3x^2+2x+8 = A(x^2+4x)+B(x+4)+Cx^2$ equation 2

Equate Coefficents Method: use equation 2 to get the following equations

x^2 |	$3 = A + C$
x |	$2 = 4A + B$
constant |	$8 = 4B$

From these equations we can see that $B = 2$, $A = 0$ and $C = 3$.

Evaluation Method: use equation 1 to get the following.

evaluate $x = 0$ to get $8 = 4B$ or $B = 2$

evaluate $x = -4$ to get $48 = 16C$ or $C = 3$

To solve for A either plug in another value for x or use the equations in the Equate Coefficents Method to find that $A = 0$.

$$\int\left(1+\frac{3x^2+2x+8}{x^2(x+4)}\right)dx = \int\left(1+\frac{2}{x^2}+\frac{3}{x+4}\right)dx = x-\frac{2}{x}+3\ln|x+4|+C$$

5. Since the degree of the denominator is larger than the degree of the numerator, long division is not needed.

$$\frac{2x+1}{(x+3)(x^2+1)^2} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\frac{2x+1}{(x+3)(x^2+1)^2} = \frac{A(x^2+1)^2}{(x+3)(x^2+1)^2} + \frac{(Bx+C)(x+3)(x^2+1)}{(x^2+1)(x+3)(x^2+1)} + \frac{(Dx+E)(x+3)}{(x^2+1)^2(x+3)}$$

$$2x+1 = A(x^2+1)^2 + (Bx+C)(x+3)(x^2+1) + (Dx+E)(x+3) \qquad \text{equation 1}$$

$$2x+1 = A(x^4+2x^2+1) + (Bx+C)(x^3+3x^2+x+3) + (Dx+E)(x+3)$$

$$2x+1 = A(x^4+2x^2+1) + B(x^4+3x^3+x^2+3x)+$$

$$C(x^3+3x^2+x+3) + D(x^2+3x) + E(x+3) \qquad \text{equation 2}$$

Evaluation Method: use equation 1 to get the following.

The only number that will allow us to quickly solve for a constant is $x = -3$. Using equation 1 gives $-5 = A(10)^2$ or $A = \frac{-1}{20}$

Equate Coefficents Method: use equation 2 to get the following.

x^4 |	$0 = A + B$	equation 3
x^3 |	$0 = 3B + C$	equation 4
x^2 |	$0 = 2A + B + 3C + D$	equation 5
x |	$2 = 3B + C + 3D + E$	equation 6
constant |	$1 = A + 3C + 3E$	equation 7

Using the value of A that was found with the Evaluation method, we can quickly solve equation 3 to find $B = \frac{1}{20}$. Using this value and equation 4 we fget that $C = \frac{-3}{20}$. Now these values combined with equation 5 and equation 7 gives $D = \frac{1}{2}$ and $E = \frac{1}{2}$.

$$\int \frac{2x+1}{(x+3)(x^2+1)^2}\,dx = \int \left(\frac{\frac{-1}{20}}{x+3} + \frac{\frac{1}{20}x + \frac{3}{20}}{x^2+1} + \frac{\frac{1}{20}x + \frac{1}{20}}{(x^2+1)^2} \right) dx$$

$$= \int \frac{-1}{20}\frac{1}{x+3}\,dx + \underbrace{\int \frac{1}{20}\frac{x}{x^2+1}\,dx}_{\text{u-sub: } u=x^2+1} + \int \frac{3}{20}\frac{1}{x^2+1}\,dx + \underbrace{\int \frac{1}{20}\frac{x}{(x^2+1)^2}\,dx}_{\text{u-sub: } u=x^2+1} + \underbrace{\int \frac{1}{20}\frac{1}{(x^2+1)^2}\,dx}_{\text{trig-sub: } x=\tan\theta}$$

for the trig sub work see number 3 in the 7.3 additional problems.

Answer: $\frac{-1}{20}\ln|x+3| + \frac{1}{40}\ln|x^2+1| + \frac{3}{20}\arctan(x) - \frac{1}{4(x^2+1)} + \frac{1}{2}\left[\frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)}\right] + C$