Section 11.1: Additional Problems Solutions

1. we can rewrite the sequence with all terms being fractions, after we notice the pattern.
$\left\{1, \frac{6}{4}, \frac{9}{5}, 2, \frac{15}{7}, ...\right\} = \left\{\frac{3}{3}, \frac{6}{4}, \frac{9}{5}, \frac{12}{6}, \frac{15}{7}, ...\right\}$
Notice the numerator is increasing by three each term so if n starts at one we can use the formula $3n$ to generate the numbers in the numerator.

   The denominator increases by one each term so if n starts at one we can use the formula $n+2$ to generate the numbers in the numerator. Another method of getting this formula is to notice the constant rate of change, which means the formula is a line. The first point is n=1 value = 3, i.e. (n,y) = (1,3), and the second point is n=2 value is 4. now find the equation of the line to get $y = n+2$.

   $a_n = \frac{3n}{n+2}$ with n starting at 1.

2. we can rewrite the sequence with all terms being fractions, after we notice the pattern.
$\left\{\frac{-1}{3}, \frac{2}{5}, \frac{5}{7}, \frac{8}{9}, 1, \frac{14}{13}, ...\right\} = \left\{\frac{-1}{3}, \frac{2}{5}, \frac{5}{7}, \frac{8}{9}, \frac{11}{11}, \frac{14}{13}, ...\right\}$
Notice that numerator and the denominator both have a constant rate of change of the values. So use a line to find the formula in the numerator and denominator.

   Numerator points(n, y) are $(1, -1)$ and $(2, 2)$ gives the equation $y = 3n - 4$

   Denominator points (n, y) are $(1, 3)$ and $(1, 5)$ gives the equations $y = 2n + 1$

   $a_n = \frac{3n-4}{2n+1}$

3. $\lim_{n\to\infty} \arcsin\left(\frac{2n}{4n+5}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ since $\lim_{n\to\infty} \frac{2n}{4n+5} \overset{L'H}{=} \frac{1}{2}$

   Thus the sequence converges and it will converge to the value $\frac{\pi}{6}$.

4. since $\frac{\pi+4}{e^2} \approx 0.9665 < 1$ we know that $\lim_{n\to\infty} \left[5 - \left(\frac{\pi+4}{e^2}\right)^n\right] = 5 - 0 = 5$

   Thus the sequence converges and it will converge to the value 0.

5. $\lim_{n\to\infty} \left[\frac{n^2}{2n-1} - \frac{n^2}{2n+1}\right] = \lim_{n\to\infty} \frac{n^2(2n+1) - n^2(2n-1)}{(2n-1)(2n+1)} = \lim_{n\to\infty} \frac{2n^2}{4n^2-1} \overset{L'H}{=} \frac{2}{4} = \frac{1}{2}$

   Thus the sequence converges and it will converge to the value $\frac{1}{2}$.

6. $\lim_{n\to\infty} \sqrt[n]{3^{2n+5}} = \lim_{n\to\infty} 3^{(2n+5)/n} = 3^2 = 9$ since $\lim_{n\to\infty} \frac{2n+5}{n} \overset{L'H}{=} 2$

   Thus the sequence converges and it will converge to the value 9.

7. Since $\lim_{n\to\infty} a_n = 7$ we know the sequence will converge to 7 and thus the sequence is bounded.

   The sequence is embeded in the function $f(x) = 7 - \frac{4}{x^2}$ and $f'(x) = \frac{8}{x^3} > 0$ for $x > 0$. Thus $f(x)$ is an increasing function for $x > 0$.

   Thus the sequence is both an increasing sequence for $x > 0$ and the sequence is bounded.

8. Consider the function $f(x) = \frac{\sqrt{x-3}}{5x+8}$. If this function is decreasing for $x \geq 3$ then the sequence, which is a part of this function will be decreasing.

   $$f'(x) = \frac{(5x+8) * \frac{1}{2}(x-3)^{-1/2} - 5 * \sqrt{x-3}}{(5x+8)^2} = \frac{(5x+8) * \frac{1}{2}(x-3)^{-1/2} - 5 * \sqrt{x-3}}{(5x+8)^2} * \frac{2\sqrt{x-3}}{2\sqrt{x-3}}$$

   note: the blue fraction is a quick way to simplify the negative exponents in the fraction.

   $$f'(x) = \frac{(5x+8) - 10 * (x-3)}{2(5x+8)^2\sqrt{x-3}} = \frac{5x + 8 - 10x + 30}{2(5x+8)^2\sqrt{x-3}} = \frac{38 - 5x}{2(5x+8)^2\sqrt{x-3}}$$

   Setting $f'(x) = 0$ and solving for x gives $x = \frac{38}{5} = 7.6$. Now looking at a sign chart for the first derivative, we see that the first derivative is negative for $x > 7.6$ and thus the function $f(x)$ is decreasing for $x \geq 8$.

   $f'(x)$: x x | 3 | + | 7.6 | $-$

9. $a_n = \frac{(-5)^n}{2^{3n}} = \frac{(-1)^n 5^n}{(2^3)^n} = \frac{(-1)^n 5^n}{8^n} = (-1)^n \left(\frac{5}{8}\right)^n$. Since $\lim_{n\to\infty} \left(\frac{5}{8}\right)^n = 0$, we know that this sequence will converge to 0.

10. Since $\lim_{n\to\infty} \frac{n^2+1}{3n^3+5} \overset{L'H}{=} 0$ and $\lim_{n\to\infty} \frac{3n}{5n+7} \overset{L'H}{=} \frac{3}{5}$

    $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{(-1)^n(n^2+1)}{3n^3+5} + \frac{3n}{5n+7} \right) = 0 + \frac{3}{5} = \frac{3}{5}$$

    Thus the sequence converges and it converges to the value $\frac{3}{5}$

11. Since $\lim_{n\to\infty} \frac{2^n}{n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{2^n\ ln(2)}{2n} \overset{L'H}{=} \lim_{n\to\infty} \frac{2^n\ (ln(2))^2}{2} = \infty$, this sequence will diverge.

12. Since $\left\{\frac{(-3)^n}{n!}\right\} = \left\{\frac{(-)^n 3^n}{n!}\right\}$ and the fact the terms of the sequence have alternating signs, we can apply the squeeze theorem.

$-b_n = \frac{-3^n}{n!} \leq \frac{(-)^n 3^n}{n!} \leq \frac{3^n}{n!} = b_n$

Now lets look at b_n for $n > 3$ we get

$$0 < b_n = \overbrace{\frac{3}{1} * \frac{3}{2} * \frac{3}{3} * \underbrace{\frac{3}{4} \cdots \frac{3}{n-1}}_{\text{each fraction is } <1} * \frac{3}{n}}^{\text{n}-\text{terms}} < \frac{3}{1} * \frac{3}{2} * \frac{3}{3} * \frac{3}{n} = \frac{27}{2n}$$

note: if you multiply a number, say J, by a positive fraction less than one, then the result will be smaller than J.

Since $\lim_{n\to\infty} \frac{27}{2n} = 0$ we get that b_n will converge to 0. Also $-b_n$ will converge to 0 as $n \to \infty$.

Since b_n and $-b_n$ both converge to 0, we know by the squeeze threorm that $\left\{\frac{(-3)^n}{n!}\right\}$ will converge to 0.

13. Since the sequence is decreasing and bounded, we know that the sequence will converge. Thus we can assume that $\lim_{n\to\infty} a_n = L$. we also get that $\lim_{n\to\infty} a_{n+1} = L$.

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{1}{3-a_n} \\ L &= \frac{1}{3-L} \\ L(3-L) &= 1 \\ L^2 - 3L + 1 &= 0 \end{aligned}$$

By the quadratic formula we get that $L = \frac{3-\sqrt{5}}{2} \approx 0.382$ or $L = \frac{3+\sqrt{5}}{2} \approx 2.618$

Since the first term of the sequence is 2 and the sequence is decreasing, we see that the limit is $L = \frac{3-\sqrt{5}}{2}$

14. Since we are told to assume that the sequence converges., we have $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_{n+1} = L$.

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{(a_n)^2+6}{4a_n} \\ L &= \frac{L^2+6}{4L} \\ 4L^2 &= L^2+6 \\ 3L^2 &= 6 \\ L^2 &= 2 \end{aligned}$$

so $L = -\sqrt{2}$ or $L = \sqrt{2}$. Since we were not told anything about increasing or decresing we need to think about the terms of the sequence. Notice with the formula , $a_{n+1} = \frac{(a_n)^2+6}{4a_n}$, the numerator will always be positive and the sign of the denominator will depend on the sign of the value we plug in. Thus if we start with a negative value, then the next term will also be negative. Likewise, if we start with a positive value, then the next term will be positive.

Since $a_1 = 3$, which is positive, we know all terms of the sequence are positive and thus the sequence converges to $L = \sqrt{2}$