Section 11.5: Additional Problems Solutions

2) Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, where $b_n = \frac{1}{n}$ if n is odd and $b_n = \frac{1}{n^2}$ is even, is divergent. Why does the alternating series Test not apply?

First figure out what the series looks like.

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \cdots$$

Now lets consider the following terms as a series.

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \sum_{n=1}^{\infty}\frac{1}{4n^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

This series is a p-series with $p = 2$ and thus it converges.

From section 11.2 we know if we have two convergent series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$, then $\sum_{n=1}^{\infty} a_n + c_n$ will also be convergent.

So lets <u>assume</u> that $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \cdots$ is a convergent series.

By the above 11.2 information then

$\sum_{n=1}^{\infty}(-1)^{n-1}b_n + \sum_{n=1}^{\infty}\frac{1}{(2n)^2}$ will also be a convergent series.

But

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n + \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \sum_{n=1}^{\infty}\frac{1}{2n-1}$$

It can be shown that $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ is a divergent series, using the limit comparison test with the harmonic series. Thus our assumption is incorrect.

$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \cdots$ is a divergent series.

Notice the alternating series does not apply since the b_n terms are not decreasing.