2) Find a power series for the function $f(x)$.

$f(x) = x^2 \ln(1+x^2)$

Known Fact:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

with $|x|<1$

Method #1

consider $g(x) = \ln(1+x^2)$

$$g'(x) = \frac{2x}{1+x^2} = 2x\left(\frac{1}{1+x^2}\right)$$

we know $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0} (-x^2)^n$ with $|-x^2|<1$

ie $|x|<\sqrt{1}=1$

ie $R=1$

$$= \sum_{n=0} (-1)^n x^{2n}$$

Thus $g'(x) = 2x \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$ with $|x|<1$

Now Integrate $g'(x)$

$$g(x) = C + \int g'(x)\,dx = C + \int \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+2}}{2n+2}$$

Since $g(x) = \ln(1+x^2)$

Then $g(0) = \ln(1+0) = \ln(1) = 0$

and $0 = C + \sum \frac{(-1)^n \cdot 2(0)^{2n+2}}{2n+2} = C + 0$

Thus $C = 0$

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Thus $g(x) = \sum_{n=0} \frac{(-1)^n 2 x^{2n+2}}{2n+2} = \ln(1+x^2)$

$f(x) = x^2 \ln(1+x^2) = x^2 g(x)$

$= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 2 x^{2n+2}}{2n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2 \cdot x^{2n+4}}{2(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{n+1}$

note the Radius of convergence is $R = 1$

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Method #2

we know $\ln|1+x| = \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1}$

so $\ln(1+x^2) = \sum_{n=0} \frac{(-1)^n (x^2)^{n+1}}{n+1}$

$= \sum_{n=0} \frac{(-1)^n x^{2n+2}}{n+1}$

Thus $x\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{n+1}$