Real Variables
Named Theorems

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1 (Well-ordering Theorem) Given any set A, there is a well-order on A

2 (Axiom of Choice) If $\{X_t \mid t \in I\}$ is a family of non-empty sets then $\Pi_{t\in I} X_t \neq 0$ where $\Pi_{t\in I} X_t = \{f : I \to \cup_{t\in I} X_t \mid \forall t \in I, f(A) \in X_t\}$.

3 (Cantor-Schrïder-Bernstein) If $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A)$ then $\text{card}(A) = \text{card}(B)$.

4 (Zorn's Lemma) Assume $(X, \leq)$ is a partially ordered set. Assume every limiting order subset (i.e. chain) of X has an upper bound. Then X has a maximal element.

5 (Hausdorff Maximal Principle) Let $(X, \leq)$ be a partially ordered set. Then there exists a maximal chain in X

i.e. if $Y \subseteq X$ such that $(Y, \leq)$ is linearly ordered and if $Z \subseteq X$ with Z linearly ordered and $Z \supseteq Y$ then $Z = Y$.

6 (Caratheodory) Suppose μ^* is an outer measure on X and set $\mathcal{M} = \mathcal{M}_{\mu^*}$ = all μ^*-measurable subsets of X. Then $\mathcal{M}$ is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

7 (monotone convergence) If $0 \leq f_1 \leq f_2 \leq \ldots$ with $f_n \in L^+$ and $f = \lim_n f_n$ pointwise, then $\int f_n d\mu \to \int f d\mu$.

8 (Fatou's Lemma) For $f_n \in L^+$ then

$$\int \liminf f_n \leq \liminf \int f_n$$

9 (Dominated Convergence Theorem, v1) If $0 \leq f_n \leq g$ are all measurable and $f_n \to_X f$, $\int g < \infty$ then $\int f_n \to \int f$.

10 (Dini's Theorem) For $f_n \in \mathcal{C}([0,1])$, $f_1 \geq f_2 \geq \dots$, $f_n \to_{[0,1]} 0$ then f_n converges to 0 uniformly on $[0,1]$.

11 (Generalized Dominated Convergence Theorem) Let $g, g_n \in L^+$ be measurable, $|f_n| \leq g_n$ μ-a.e., $f_n \to f$ and $g_n \to g$ μ-a.e. with $\int g_n \to \int g < \infty$.

Then $\int f_n \to \int f$. Moreover, $\int |f - f_n| \to 0$

12 (Egoroff's Theorem) Suppose $f_n \to f$ a.e. and $\mu(D) < \infty$. Then $\chi_D f_n \to \chi_D f$ almost uniformly.

13 (types of convergence)

(A) $f_n \rightrightarrows f$ (uniform)

i.e. $\|f_n - f\|_{\sup} \to 0$

(B) $f_n \to f$ pointwise

i.e. $f_n(x) \to f(x)$ for all x

(C) $f_n \to f$ a.e.

i.e. $\mu(\{x \mid f_n(x) \not\to f(x)\}) = 0$ (this is not a topological mode of convergence)

(D) $f_n \to f$ (μ) (in measure)

i.e. $\forall \epsilon > 0$, $\lim_n \mu[|f - f_n| > \epsilon] = 0$

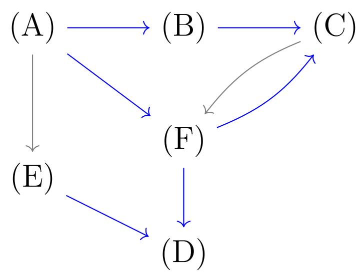
(E) $L^1(\mu)$ convergence

i.e. $\|f_n - f\|_1 \to 0$

(F) $f_n \to f$ almost uniformly

i.e. $\forall \epsilon > 0$, $\exists E$ such that $\mu(E^C) < \epsilon$ and $f_n \rightrightarrows_E f$

The following diagram shows the implications where blue arrows mean on any measure space and gray arrows mean it only holds on finite measure spaces.



(F) $\not\to$ (E) and (D) $\to$ (C) for a subsequence.

(C) or (D) + (dominated or monotonicity) $\to$ (E)

$f_n \to f$ in L^1 $\Leftrightarrow$ every subsequence of f_n has a further subsequence which converges to f in L^1.

14 (Tonelli) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and $f : X \times Y \to [0, \infty]$ be a measurable function. Then

1. Define $f_x : Y \to [0, \infty]$ by $y \mapsto f(x, y)$. Then f_x is measurable for all $x \in X$
2. $x \mapsto \int f(x, y) d\nu(y)$ is a measurable function on X
3. $\int f d(\mu \times \nu) = \int (\int f(x, y) d\nu(y)) d\mu(x)$

15 (Fubini) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, $f \in L^1(\mu \times \nu)$. Then

1. for μ-a.e. $x \in X$, $f(x, \cdot) \in L^1(\nu)$
2. $x \mapsto \int_Y f(x, y) d\nu(y) \in L^1(\mu)$
3. $\int f d\mu \times \nu = \int (\int f(x, y) d\nu(y)) d\mu(x)$

If f is measurable on $X \times Y$ then $|f|$ is measurable on $X \times Y$.

16 (Approximation properties of m^n) We let m^n be the completion of $m \times \cdots \times m$ where m is the Lebesgue measure on $\mathbb{R}$. So $\mathcal{L}^n$ is the Lebesgue measurable sets on $\mathbb{R}^n$.

Take $E \in \mathcal{L}^n$. Then

1. $m^n(E) = \inf\{m^n(\mathcal{O}) \mid E \subseteq \mathcal{O} \text{ and } \mathcal{O} \text{ is open}\}$

   $= \sup\{m^n(K) \mid K \subseteq E \text{ and } K \text{ is compact}\}$.
2. $E = A_1 \backslash N_1$ where A_1 is G_δ and $m^n(N_1) = 0$

   $E = A_2 \cup N_2$ where A_2 is F_σ and $m^n(N_2) = 0$
3. $m^n(E) < \infty$ implies $\forall \epsilon > 0$, $\exists (R_j)_{j=1}^N$ of disjoint open rectangles such that $m^n(E \triangle (\cup R_j)) = 0$

17 (Hahn-Decomposition Theorem) Let ν be a signed measure on $(X, \mathcal{M})$. Then there exists $P \in \mathcal{M}$ which is positive for ν and $N = P^C$ is negative for ν.

Moreover, the decomposition $X = P \cup N$ is essentially unique: if P_1 is positive for ν and $N_1 = P_1^C$ is negative for ν, then $P \triangle P_1 = N \triangle N_1$ is null for ν.

18 (Jordan Decomposition) Take the Hahn decomposition and let $\nu^+(E) := \nu(E \cap P)$, $\nu^-(E) := -\nu(E \cap N)$ so that $\nu = \nu^+ - \nu^-$. Note that $\nu^+ \perp \nu^-$.

Note that the Jordan Decomposition is unique.

The total variation of ν is defined to be $|\nu|(E) = \nu^+(E) + \nu^-(E)$.

19 (Lebesgue Decomposition Theorem) Let μ be a measure on $(X, \mathcal{M})$ and ν a σ-finite signed measure. Then $\nu = \nu_1 + \nu_2$ where $\nu_1 \perp \mu$, $\nu_2 \ll \mu$. Moreover, this decomposition is unique.

20 (Radon-Nikodyn Theorem) If $(X, \mathcal{M})$ is a measurable space, μ a σ-finite measure on $\mathcal{M}$ and ν a σ-finite signed measure on $\mathcal{M}$ with $\nu \ll \mu$, then there exists an extended μ-integrable f such that $\nu = \nu_f$ where $\nu_f(E) = \int_E f d\mu$.

Moreover, we have uniqueness. If $\nu_f = \nu_g$ then $f = g$ μ-a.e.

21 (Lebesgue Differential Theorem) Fix $x \in \mathbb{R}^n$. We say $\{E_r\} \subseteq \mathcal{B}_{\mathbb{R}^n}$ shrinks nicely to x if

- $E_r \subseteq B(r, x) \qquad \forall r > 0$
- $\exists \alpha > 0$ such that $\forall r > 0$, $m(E_r) \geq \alpha m(B(r, x))$

Lebesgue Differential Theorem: For $f \in L^1_{loc}(\mathbb{R}^n)$, then for all $x \in L_f$ and for all $\{E_r\}$ shrinking nicely to x, we have

$$\lim_{r \to 0^+} \frac{\int_{E_r} |f(y) - f(x)| dy}{m(E_r)} = 0$$

$$f(x) = \lim_{r \to 0^+} \frac{\int_{E_r} f(y) dy}{m(E_r)}$$

22 (Urysohn's Lemma) Let $(X, \mathcal{T})$ be normal. If A, B are disjoint closed sets and $a \neq b$ in $\mathbb{R}$. Then there exists some $f \in C(X, [a, b])$ such that $f|_A \equiv a$ and $f|_B \equiv b$.

proof uses nastay lemma

23 (Tiktze Theorem) **Version 1:** Let $(X, \mathcal{T})$ be normal. If $A \subseteq X$ is closed and $f \in C(A, (a, b))$ then there exists some $F \in C(X, [a, b])$ such that $F|_A = f$.

Version 2: Let $(X, \mathcal{T})$ be normal. If $A \subseteq X$ is closed and $f \in C(A, (a, b))$ then there exists some $F \in C(X, \mathbb{R})$ such that $F|_A = f$.

24 (Tychonoff Theorem) If (X_α) are compact topological spaces, then $X = \Pi_{\alpha \in \mathcal{A}} X_\alpha$ (with the product topology) is compact.

Theorem: Axiom of Choice $\Leftrightarrow$ Tychonoff

25 (Arzela-Ascoli) We say a metric space X is totally bounded if for any $r > 0$, X can be covered by a finite number of balls of radius r.

Arzela-Ascoli Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $\mathcal{C}(X)$ then $\mathcal{F}$ is totally bounded in the uniform metric and the closure of $\mathcal{F}$ in $\mathcal{C}(X)$ is compact.

Alternative version 1: Let X be a σ-compact LCH space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $\mathcal{C}(X)$, then there exists a $f \in \mathcal{C}(X)$ and a subsequence of

$\{f_n\}$ that converges to f uniformly on compact sets.

Alternative version 2: Let X be compact and $\mathcal{F} \subseteq \mathcal{C}(X)$. Then $\overline{\mathcal{F}}$ is compact in $\mathcal{C}(X)$ IFF

1. $\mathcal{F}$ is equicontinuous
2. $\mathcal{F}$ is pointwise bounded

26 (Stone-Weierstrass) $\mathcal{A}$ is called an algebra if it is a real vector subspace of $C(X)$ such that $fg \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$.

Let X be a compact, Hausdorff space and $\mathcal{B} \subseteq \mathcal{C}(X, \mathbb{R})$ a subalgebra such that $\mathcal{B}$ separates points (that is, for $x \neq y, \exists f \in \mathcal{B}$ with $f(x) \neq f(y)$). Then if there exists some $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{B}$, then $\overline{\mathcal{B}} = \{f \in \mathcal{C}(X, \mathbb{R}) \mid f(x_0) = 0\}$. Otherwise, $\overline{B} = \mathcal{C}(X)$.

27 (Hahn-Banach) For a real vector space X, we say $p : X \to \mathbb{R}$ is a sublinear mapping if $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ when $\lambda \geq 0$.

Hahn-Banach: Let X be a real vector space, p a sublinear functional on X, M a subspace of X, and f a linear functional on M such that $f|_M \leq p|_M$. Then there exists a linear functional F on X such that $F \leq p$ on X and $F|_M = f$.

For the complex case, we require $|f(x)| \leq p(x)$ and we get $|F(x)| \leq p(x)$.

28 (Baire Category) We say C is nowhere dense if $(\overline{C})^\circ = \emptyset$.

Theorem: Let X be a complete metric space. Then if $\{U_n\}$ is a sequence of open dense sets, $\cap U_n$ is dense. Thus, X is not a countable union of nowhere dense sets.

A set that is a countable union of nowhere dense sets is said to be of first category (and it's complement is called residual). A set which is not a countable union of nowhere dense sets is called second category.

29 (uniform boundedness principle) Let X be a Banach space and Y a normed space, $\mathcal{S} \subseteq L(X, Y)$ where $\mathcal{S}$ is pointwise bounded (i.e. $\forall x \in X, \sup\{\|Tx\| \mid T \in \mathcal{S}\} < \infty$).

Then $\mathcal{S}$ is uniformly bounded (i.e. $\sup_{T \in \mathcal{S}} \|T\| < \infty$.

30 (Banach-Steinhaus) Suppose X is a Banach space and Y is a normed space, and $\{T_n\} \subseteq L(X, Y)$ and for all $x \in X$, $T_n x \to Tx$ in Y. Then $T \in L(X, Y)$.

31 (open mapping theorem) **little open mapping theorem:** Suppose X is a Banach space and Y is a normed space, $T \in L(X, Y)$ and $r > 0$. Then if $\overline{T(B(0,1))} \supseteq B(0, r)$ then $T(B(0, 1)) \supseteq B(0, r)$.

open mapping theorem: Suppose X, Y are Banach spaces and $T \in L(X, Y)$ is surjective. Then T is an open mapping.

Remark: For a linear map T, T is open $\Leftrightarrow \exists r>0$ such that $T(B(0,1)) \supseteq B(0,r)$.

32 (closed graph) For Banach spaces X, Y and $T : X \to Y$ linear, then $T \subseteq X \times Y$ is closed $\Leftrightarrow T$ is a bounded linear operator.

33 (Separation Theorem / Geometric Hahn-Banach) Say X is a LCTVS over $\mathbb{R}$ and $U, C \subseteq X$ are convex sets such that $U \cap C = \emptyset$ and $U^\circ \neq \emptyset$. Then there exists some non-zero $f \in X^*$ and some $\alpha \in \mathbb{R}$ such that $U \subseteq [f < \alpha]$ and $C \subseteq [f \geq \alpha]$

Corollary 1: If $(X, \mathcal{T})$ is Hausdorff LCTVS, then X^* separates points of X

Corollary 2: If $(X, \mathcal{T})$ is a LCTVS, $C \subseteq X$ is convex, then $\overline{C}^{\text{weak}} = \overline{C}^{\mathcal{T}}$.

Corollary 3: If X is a normed space and $A \subseteq X$, then A is norm bounded $\Leftrightarrow A$ is weakly bounded (where A is weakly bounded if for all $x^* \in X^*$, $\sup_{x \in X} |\langle x^*, x\rangle| < \infty$

34 (Banach-Alaoglu) If X is a normed space, then $\overline{B_{X^*}} = \{x^* \in X^* \mid \|x^*\| \leq 1\}$ is weak*-compact.

Corollary: If X is reflexive, then $\overline{B_{X^*}}$ is weakly compact.

X is reflexive if and only if $\overline{B_X}$ is weakly compact.

35 (Goldstine) Suppose X is normed. Then $\widehat{B_X}$ is weak*-dense in $B_{X^{**}}$, $\widehat{B_X} \subseteq B_{X^{**}}$ where we have equality IFF X is reflexive.

36 (Riesz-Fisher) For $1 \leq p < \infty$, L^p is complete

37 (Hölder's inequality) Let q be the conjugate exponent of p so $\frac{1}{p} + \frac{1}{q} = 1$ (i.e. $q = \frac{p}{p-1}$)

For measurable f, g and $1 < p < \infty$ then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

If $f \in L^p$ and $g \in L^q$ if and only if $f = 0$ a.e. OR $g = 0$ a.e. OR $|f|^p$ is a scalar multiple of $|g|^q$.

If $f \in L^p$ then $\|f\|_p = \max\left\{\int fg d\mu \mid \|g\|_q \leq 1\right\}$ (maximum is achieved! by $g = \text{sgn}(f)$).

Alternate Hölder's inequality: For $0 < \lambda < 1$, then $\int |f|^\lambda |g|^{1-\lambda} \leq \left(\int |f|\right)^\lambda \left(\int |g|\right)^{1-\lambda}$.

38 (Minkowski) For $1 \leq p < \infty$, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

39 (Riesz-Thorin) If $1 \leq p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$ and $0 < t < 1$ with

$$\frac{1}{p_t} := \frac{t}{p_0} + \frac{1-t}{p_1} \qquad \frac{1}{q_t} := \frac{t}{q_0} + \frac{1-t}{q_1}$$

Suppose $X_0 = L^{p_0}(\mu)$, $X_0 = L^{p_1}(\mu)$ and $Y_0 = L^{q_0}(\nu)$, $Y_1 = L^{q_1}(\nu)$ (compatible couple).

Then for $0 < t < 1$, $L^{p_t}(\mu) + L^{q_t}(\nu)$ is an exact interpolation pair for $\widetilde{X} = (X_0, X_1), \widetilde{Y} = (Y_0, Y_1)$.

40 (Marcinkiewicz Interpolation) Let $(X, \mathcal{M}, \mu)$ be a measure space and D a subspace of $L^0(\mu)$. We say $T : D \to L^0(\nu)$ is sublinear if

1. $|T(f+g)| \leq |Tf| + |Tg|$
2. $|T(cf)| = c|Tf|$ if $c \geq 0$

T is said to be of strong type (p,q) if $T(L^p(\mu)) \subseteq L^q(\nu)$ and $\|T|_{L^p(\mu)}\|_{L^p(\mu)\to L^q(\nu)} < \infty$.

T is said to be of weak type (p,q) if $T(L^p(\mu)) \subseteq L^{q,\infty}(\nu)$ and $\|T|_{L^p(\mu)}\|_{L^p(\mu)\to L^{q,\infty}(\nu)} =: \sup_{\|x\|_{L^p(\mu)}\leq 1}[Tx]_{q,\infty} < \infty$ where for $q < \infty$, $L^{q,\infty}(\nu) = \{f \in L^0(\nu) \mid \sup_t t^{1/q}\nu[|f| > t] =: [f]_{q,\infty} < \infty\}$.

Weak type (p, ∞) is the same as strong type (p, ∞).

Marcinkiewicz Interpolation Theorem: $1 \leq p_0 \leq q_0 \leq \infty$ and $1 \leq p_1 \leq q_1 \leq \infty$, $q_0 \neq q_1$ and $0 < t < 1$,

$$\frac{1}{p_t} := \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q_t} := \frac{1-t}{q_0} + \frac{t}{q_1}$$

If $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to L^0(\nu)$ is sublinear, and is of weak type $(p_0, q_0$ and weak type (p_1, q_1) then T is of strong type (p_t, q_t) for all $0 < t < 1$ and

$$\|T\|_{L^{p_t}\to L^{q_t}} \leq \frac{C\Big(\|T\|_{L^{p_0}\to L^{p_0,\infty}} \vee \|T\|_{L^{p_1}\to L^{p_1,\infty}}\Big)}{t(1-t)}$$

where $C = C(p_0, p_1, q_0, q_1)$ is some constant $< \infty$.

41 (Krein-Milman) If C is a convex set in a real vector space, then $x \in C$ is said to be an extreme point provided whenever $y, z \in C$ and $0 < \lambda < 1$, $x = \lambda y + (1-\lambda)z$ then $x = y = z$.

Krein-Milman Lemma: If X is a Hausdorff LCTVS, and $C \subseteq X$ is a non-empty, compact, convex set then $\text{ext}(C) \neq \emptyset$.

Krein-Milman Theorem: If X is a Hausdorff LCTVS, $C \subseteq X$ is a non-empty, compact, convex set, then $C = \overline{\text{conv}(\text{ext}(C))}$, where $\text{ext}(C) = \{$ all extreme points of C $\}$.

42 (Banach-Stone) Suppose K_1, K_2 are compact Hausdorff. Then $C(K_1)$ is isometrically isomorphic to $C(K_2)$ if and only if K_1 is homeomorphic to K_2.

43 (Milman) If X is Hausdorff LCTVS and $M \subseteq X$ is compact with $C = \overline{\text{conv}(M)}$

compact. Then $\text{ext}(C) \subseteq M$.

44 (Kakatani fixed point theorem) We say T is an affine transformation if $T(\alpha x + (1-\alpha)y) = \alpha Tx + (1-\alpha)Ty$ for $0 \leq \alpha \leq 1$, $x, y \in K$.

G is equicontinuous if for all neighborhoods U of 0, there exists a neighborhood V of 0 such that for $x, y \in K$, if $x - y \in V$ then for all $T \in G$, $Tx - Ty \in U$.

We call p a fixed point of G if $G(p) = \{Tp \mid T \in G\} = \{p\}$.

Theorem: Suppose X is a LCTVS and $K \subseteq X$ is convex compact, and G is an equicontinuous group (under composition) of affine transformations on K. Then G has a fixed point.