michael4236@tamu.edu
3D COORDINATE SYSTEMS

- Rectangular
- Cylindrical \((r, \theta, z)\)
- Spherical (not limited to just spheres) \((\rho, \theta, \phi)\)
The distance formula is:

2D: \( \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \)

3D: \( \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} \)

4D: \( \sqrt{\ldots + (w_1 - w_0)^2} \)
Example: Find the distance from the point \( P(2, -1, 7) \) to the point \( Q(1, -3, 5) \)

\[
\overrightarrow{PQ} = \sqrt{(1 - 2)^2 + (-3 - (-1))^2 + (5 - 7)^2}
\]

\[
= \sqrt{(-1)^2 + (-2)^2 + (-2)^2}
\]

\[
= \sqrt{1 + 4 + 4} = \sqrt{9} = 3
\]
CYLINDRICAL COORDINATES

- Cylindrical coordinates are of the form \((r, \theta, z)\) where \(\theta\) can be found using \(\text{arctan}\) as in polar coordinates.

- What does the equation \(x^2 + y^2 = 1\) represent as a surface on the surface \(R^3\)?
EXAMPLE

- Convert this point from cylindrical coordinates \((4, \frac{2\pi}{3}, -2)\) to rectangular coordinates

\[
\begin{align*}
 x &= r \cos \theta = 4 \cos \left(\frac{2\pi}{3}\right) = 4 \cdot -\frac{1}{2} = -2 \\
 y &= r \sin \theta = 4 \sin \left(\frac{2\pi}{3}\right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} \\
 z &= -2
\end{align*}
\]

\((-2, 2\sqrt{3}, -2)\)
SPHERICAL COORDINATES

- Spherical coordinates are of the form \((\rho, \theta, \varphi)\)
- Key note:
  - \(0 \leq \varphi \leq \pi\)
  - Spherical Coordinates are used to identify cones and spheres

\[ x = \rho \sin \varphi \cos \theta \]
\[ y = \rho \sin \varphi \sin \theta \]
\[ z = \rho \cos \varphi \]

\[ \rho^2 = x^2 + y^2 + z^2 \]
\[ \tan \theta = \frac{y}{x} \]
\[ \varphi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \]
\[ r = \rho \sin \varphi \]
EXAMPLE

- Express the coordinate \((8, \frac{\pi}{3}, \frac{\pi}{6})\) as rectangular and cylindrical coordinates
• Describe the figure $r^2 + z^2 = 100$
VECTORS
VECTORS

- Vectors are unique in that they have both direction and magnitude.
- A vector is denoted in brackets as \( <a,b,c> \) where \( a = a_2 - a_1 \).
- Visually, when adding vectors, we add them **tip to tail**.
- Vectors can be multiplied by scalar multiples.
• Represent the vector $a = <3, 2>$
EXAMPLE

- Find the vector represented by the directed line segment with initial point \( P \) (2,-3,4) and terminal point \((-2,1,1)\)

\[
\overrightarrow{PQ} = \langle -2 - 2, 1 - 3, 1 - 4 \rangle = \langle -4, -2, -3 \rangle
\]
• The length of a vector is defined as 3-dimensional vector \( a = \langle a_1, a_2, a_3 \rangle \)

\[
\mathbf{u} = \langle \sqrt{3}, \sqrt{3}, \frac{1}{\sqrt{3}} \rangle
\]

\[
|\mathbf{u}| = \sqrt{\left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2}
\]

\[
= \sqrt{3 + 3 + \frac{1}{3}} = \sqrt{\frac{19}{3}} = \frac{\sqrt{57}}{\sqrt{3}}
\]

\[
= \sqrt{19} = 9
\]

• The unit vector is a vector that has length 1. To convert a vector into a unit vector simply divide the components of the vector the length (a scalar value)

\[
\mathbf{u} = \frac{a}{|a|} = \frac{4a}{|a|}
\]

\[
|\mathbf{u}| = 1
\]
EXAMPLE

• Find the length of the vector $a = \langle 2, 4, 4 \rangle$

$$\left| \mathbf{a} \right| = \sqrt{2^2 + 4^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$
EXAMPLE

• Find the unit vector in the direction of the vector

\[ a = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} \]

\[ a = \langle 2, -1, -2 \rangle \]

\[ |a| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} \]

\[ = \sqrt{9} = 3 \]

\[ \mathbf{u} = \frac{a}{|a|} = \frac{\langle 2, -1, -2 \rangle}{3} = \langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \rangle \]

\[ -\mathbf{u} = \langle -\frac{8}{3}, \frac{4}{3}, \frac{8}{3} \rangle \]
If \( a = \langle 4, 0, 3 \rangle \) and \( b = \langle -2, 1, 5 \rangle \), find \(|a|\) and \(2a + 5b\).

\[
2 \langle 4, 0, 3 \rangle + 5 \langle -2, 1, 5 \rangle = \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle
\]
THE DOT PRODUCT
DOT PRODUCT

- The dot product converts vectors into scalars. The dot product is as follows:

\[ \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \]

- If \( \theta \) is the angle between the vectors \( \mathbf{a} \) and \( \mathbf{b} \) then

\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta \]

\[ \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \Rightarrow \theta = \cos^{-1} \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \right) \]
• Compute the dot product for $a = \langle 0,3,-2 \rangle, b = \langle 2,3,1 \rangle$

$$a \cdot b = 0 + 9 + (-2)$$

$$= 7$$
EXAMPLE

- Find the dot product if $||u||=4$, $||v||=10$, and $\theta = \frac{2\pi}{3}$

\[
\mathbf{u} \cdot \mathbf{v} = ||u|| \cdot ||v|| \cos \theta
\]

\[
= 4 \cdot 10 \cos \left( \frac{2\pi}{3} \right)
\]

\[
= 4 \cdot 10 \cdot -\frac{1}{2} = \sqrt{-20}
\]
EXAMPLE

• Determine the angle between the vector $a = \langle 4, 8, 10 \rangle$ and $b = \langle 9, 2, 7 \rangle$

\[
\theta = \cos^{-1}\left( \frac{a \cdot b}{|a| |b|} \right)
\]

\[
a \cdot b = 36 + 16 + 70 = 122
\]

\[
|a| = \sqrt{4^2 + 8^2 + 10^2} = \sqrt{16 + 64 + 100} = \sqrt{180}
\]

\[
|b| = \sqrt{9^2 + 2^2 + 7^2} = \sqrt{81 + 4 + 49} = \sqrt{134}
\]

\[
\theta = \cos^{-1}\left( \frac{122}{\sqrt{180} \cdot \sqrt{134}} \right)
\]