MATH 409.200
Final Exam
December 9, 2013

Name: ______________________________

ID#: ______________________________

The exam consists of 11 questions, the first 5 of which are multiple choice. The point value for a question is written next to the question number. There is a total of 100 points. No aids are permitted.

For questions 1 to 5 circle the correct answer.

1. [4] Let f and g be differentiable functions on $\mathbb{R}$ such that $f(0) = 3$, $f'(0) = 2$, and $g'(3) = 5$. Then $(g \circ f)'(0)$ is equal to

   (a) 5
   (b) 6
   (c) 10
   (d) 20

2. [4] Which of the following sets is countable?

   (a) the image of a nonconstant continuous function $f : (0,1) \to \mathbb{R}$

   (b) $\big\{ \sup A : A \subseteq \mathbb{Q} \cap [0,1] \big\}$

   (c) the set of all $x \in (0,1]$ such that the improper integral $\displaystyle\int_0^1 \frac{1}{t^x}\,dt$ diverges

   (d) the set of all constant sequences in $\mathbb{R}$

3. [4] $\lim_{x\to\infty} \dfrac{\ln(x^3+2)}{\ln(x^2+1)} =$

(a) 0

(b) $\frac{3}{2}$

(c) 2

(d) ∞

4. [4] The interval of convergence of the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^2}$ is

(a) $[0,2]$

(b) $(0,2)$

(c) $(0,2]$

(d) $[0,2)$

5. [4] Which of the following series diverges?

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^k}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k^{3/2}+\cos k}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{10k}}{k!}$

(d) $\displaystyle\sum_{k=1}^{\infty} \sqrt{\frac{k+k^{1/2}+1}{k^3+5}}$

6. [24] In each of the following 8 cases, indicate whether the given statement is true or false. No justification is necessary.

(a) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a bounded function then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ has a convergent subsequence.

(b) Let f be a differentiable function which is improperly integrable on $(0, \infty)$. Then $\lim\limits_{x\to\infty} f'(x) = 0$.

(c) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = 0$. Then f has a maximum value on $[0, \infty)$.

(d) The function $f(x) = \dfrac{e^x}{x+1}$ is integrable on $[0, 1]$.

(e) The function

$$f(x) = \begin{cases} x^2 \cos(\sin(\cos(1/x))), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$.

(f) $\lim_{n\to\infty} \dfrac{n \sin n}{e^{2n}} = 0.$

(g) If f is a nonconstant integrable function on $[0, 1]$ then $\displaystyle\int_0^1 |f(x)|\, dx > 0.$

(h) Let $\{f_n\}$ be a sequence of functions on $[0, 1]$ such that $|f_n(x)| \leq 1/n^2$ for all $n \in \mathbb{N}$ and $x \in [0, 1]$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[0, 1]$.

7. [12] (a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \displaystyle\int_{\cos x}^{3+\sin x} \cos t \, dt$. Determine f'.

(b) Evaluate the improper integral $\displaystyle\int_0^\infty \frac{e^{-x}}{e^{-x} + 3} \, dx$.

8. [12] (a) State what it means for a function on $\mathbb{R}$ to be uniformly continuous.

(b) State the completeness axiom for the real numbers.

(c) State the Mean Value Theorem.

9. [12] (a) State what it means for a sequence $\{f_n\}$ of functions to converge uniformly on a nonempty set $A \subseteq \mathbb{R}$ to a function f.

(b) Give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions on $[0,1]$ which converges pointwise on $[0,1]$ to a function which is not continuous at the point $\frac{1}{2}$.

(c) Give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions on $[0,1]$ such that

(i) $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[0,1]$, and

(ii) the sequence $\{\int_0^1 f_n(x)\,dx\}_{n=1}^{\infty}$ is bounded but does not converge.

10. [10] (a) Define what it means for a bounded function f on a closed interval $[a, b]$ to be integrable on $[a, b]$.

(b) Define the function $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & -1 \leq x \leq 0 \\ x, & 0 < x \leq 1. \end{cases}$$

Prove directly from the definition of integrability that f is integrable on $[-1, 1]$.

11. [10] Let f be a locally integrable function on $\mathbb{R}$ such that $f(x)$ converges to a real number L as $x \to \infty$. Prove that

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)\, dt = L.$$