MATH 409.501
Final Exam
May 8, 2009

Name: ______________________________

ID#: ______________________________

The exam consists of 11 questions, the first 5 of which are multiple choice. The point value for a question is written next to the question number. There is a total of 100 points. No aids are permitted.

For questions 1 to 5 circle the correct answer.

1. [4] Which of the following is false?

   (a) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
   (b) Every continuous function on $[0, 1]$ is uniformly continuous on $[0, 1]$.
   (c) Every absolutely convergent series of real numbers is convergent.
   (d) Every bounded function on $[0, 1]$ is the uniform limit of a sequence of continuous functions on $[0, 1]$.

2. [4] Define $g(x) = \int_{-1}^{x} |t|\,dt$ for all $x \in \mathbb{R}$. Which of the following is false?

   (a) $g'(0)$ exists
   (b) $g''(0)$ exists
   (c) $\int_0^1 g(x)\,dx \geq 0$
   (d) g is continuous on $[0, 1]$

3. [4] (a) For each $n \in \mathbb{N}$ define the function $f_n(x) = \cos(x/n)$. Which of the following functions f is the pointwise limit on $\mathbb{R}$ of the sequence $\{f_n\}_{n=1}^{\infty}$?

   (a) $f(x) = 1$ for all $x \in \mathbb{R}$

   (b) $f(x) = 0$ for all $x \in \mathbb{R}$

   (c) $f(x) = \cos x$ for all $x \in \mathbb{R}$

   (d) $f(x) = x$ for all $x \in \mathbb{R}$

4. [4] Let f and g be differentiable functions on $\mathbb{R}$ such that $f(0) = 2$, $f'(0) = 3$, and $g'(2) = 2$. Then $(g \circ f)'(0)$ is equal to

   (a) 0

   (b) 2

   (c) 3

   (d) 6

5. [4] Which of the following functions is continuous on $[0, 1]$?

   (a) the pointwise limit as $n \to \infty$ of the sequence of functions f_n given by $f_n(x) = x^n$

   (b) $f(x) = \begin{cases} x, & x \neq 1/2 \\ 0, & x = 1/2 \end{cases}$

   (c) $f(x) = \displaystyle\int_1^{\infty} \frac{1}{t^{x+2}}\, dt$

   (d) $f(x) = \begin{cases} 1, & \text{if the series } \sum_{k=1}^{\infty} k^{-x-1} \text{ converges} \\ 0, & \text{otherwise} \end{cases}$

6. [12] (a) State the completeness axiom for the real numbers.

(b) Let E be a nonempty subset of $\mathbb{R}$. State what it means for a function f to be uniformly continuous on E.

(c) State what it means for a series $\sum_{k=1}^{\infty} a_k$ of real numbers to converge.

(d) Let f be a function defined on an open interval I and let $a \in I$. State the definition of the limit of $f(x)$ as $x \to a$.

7. [14] Determine with explanation whether the series converges or diverges.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k+3}{k^2+2}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(3+(-1)^k)^k}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k^3-4)}{k^3-4}$

8. [14] (a) Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \int_0^{x^2} e^{t^2}\, dt$. Determine F'.

(b) Evaluate $\int_{-1}^{1} \frac{4|x|}{x^2+3}\, dx$.

(c) Does the improper integral $\int_0^{\infty} \frac{e^{-x}}{x^2+3}\, dx$ converge or diverge? Provide justification.

9. [14] (a) Find the radius of convergence and interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{\sqrt{k^2+1}} x^k.$$

(b) Give an example of a power series whose interval of convergence is $[0, 2]$.

10. [12] (a) State what it means for a sequence $\{f_n\}_{n=1}^{\infty}$ of functions on $[0, 1]$ to converge uniformly to a function f on $[0, 1]$.

(b) For each $n \in \mathbb{N}$ define the function $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = xe^{x/n}$, and define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x$. Prove that $f_n \to f$ uniformly on $[0, 1]$.

11. [14] (a) Define what it means for a bounded function f on a closed interval $[a, b]$ to be integrable on $[a, b]$.

(b) Prove directly from the definition of integrability that the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 1, & x = 0 \end{cases}$$

is integrable on $[0, 1]$.