MATH 410.501
Examination 1
February 12, 2014

Name: ______________________________

ID#: ______________________________

The exam consists of 3 questions. The point value for a question is written next to the question number. There is a total of 100 points. No aids are permitted.

1. [50] In each of the following ten cases, indicate whether the given statement is true or false. No justification is necessary.

(a) Let $\{f_n\}$ be a sequence of continuous functions which converges pointwise on $[0,1]$ to a function f. Then f is continuous.

(b) Let $\{f_n\}$ be a sequence of functions such that $|f_n(x)| \leq 10$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k^2} f_n(x)$ converges uniformly on $\mathbb{R}$.

(c) If the power series $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ both converge to the same function on $(-1, 1)$, then $a_3 = b_3$.

(d) If f is a differentiable function on $(-1, 1)$, then there is a sequence $\{a_k\}$ of real numbers such that $\sum_{k=0}^{\infty} a_k x^k$ converges to $f(x)$ for all $x \in (-1, 1)$.

(e) Let f be a differentiable function on $(0, 1)$ and g an analytic function on $(0, 1)$. Then fg is analytic on $(0, 1)$.

(f) For every $a \in \mathbb{R}$, the Taylor expansion of e^x centered at a is $\displaystyle\sum_{k=0}^{\infty} \frac{e^a}{k!}(x-a)^k$.

(g) For all real numbers $c_0, c_1, c_2, \ldots$ the power series $\displaystyle\sum_{k=1}^{\infty} c_k x^k$ and $\displaystyle\sum_{k=1}^{\infty} \frac{c_k}{k} x^k$ have the same radius of convergence.

(h) If a function is analytic on the interval $(0, 1)$, then it is continuous on $(0, 1)$.

(i) If a power series $\sum_{k=0}^{\infty} c_k x^k$ converges on closed interval $[a, b]$, then it converges uniformly on $[a, b]$.

(j) Let $f \in C^\infty(0, 1)$ and suppose that there is an $M > 0$ such that $|f^{(n)}(x)| \leq M$ for all $x \in (0, 1)$ and $n \in \mathbb{N}$. Then f is analytic on $(0, 1)$.

2. [25] (a) Let E be a nonempty subset of R. State what it means for a sequence $\{f_n\}$ of functions to converge uniformly on E to a function f.

(b) Give an example of a sequence $\{f_n\}$ of functions which converges pointwise on $[0, 1]$ but does not converge uniformly on $[0, 1]$.

(c) Prove that for every real number $r > 0$ the sequence of functions $f_n(x) = \dfrac{e^x}{1 + n + x^2}$ converges uniformly on $[-r, r]$ to the zero function.

3. [25] (a) Find the radius of convergence and interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{3^k}{5k} x^k.$$

(b) Give an example of a power series whose radius of convergence is ∞.