MATH 410.501
Examination 2
March 26, 2014

Name: ____________________

ID#: ____________________

The exam consists of 3 questions. The point value for a question is written next to the question number. There is a total of 100 points. No aids are permitted.

1. [50] In each of the following ten cases, indicate whether the given statement is true or false. No justification is necessary.

   (a) If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|$ if and only if $\mathbf{x} = t\mathbf{y}$ for some real number $t \geq 0$.

   (b) Let $\{\mathbf{x}_k\}_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^n$. Then the set $\{\mathbf{x}_k : k \in \mathbb{N}\}$ is not closed.

(c) Let $\{\mathbf{x}_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$ such that the sequence $\{\|\mathbf{x}_k\|\}_{k=1}^{\infty}$ converges in $\mathbb{R}$. Then the sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ converges.

(d) The set $\{(x, y) \in \mathbb{R}^2 : 0 < y < e^{x^2}\}$ is open in $\mathbb{R}^2$.

(e) Let $n, m \in \mathbb{N}$ and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then the operator norm $\|T\| := \sup_{\|\mathbf{x}\| \neq \mathbf{0}} \|T(\mathbf{x})||/\|\mathbf{x}\|$ is finite.

(f) Let A be a subset of $\mathbb{R}^2$ whose boundary is empty. Then A must be either empty or equal to all of $\mathbb{R}^2$.

(g) Let A be a subset of $\mathbb{R}^n$. Then its closure $\overline{A}$ is not open.

(h) The set $\{(x, y) \in \mathbb{R}^2 : y = x \text{ or } y = 2x\}$ is connected.

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^4}{x^2 + 3y^4}$ exists.

(j) Let $A \subseteq \mathbb{R}^n$. Then the closure of the interior of A is equal to A.

2. [25] (a) State what it means for a set $A \subseteq \mathbb{R}^n$ to be compact.

(b) Prove that if A and B are compact subsets of $\mathbb{R}^n$ then so is $A \cup B$.

(c) Give an example to show that the union of infinitely many compact subsets of $\mathbb{R}^2$ need not be compact. Provide justification.

3. [25] (a) Let $n, m \in \mathbb{N}$. Let $\mathbf{a}$ be a vector in $\mathbb{R}^n$, U an open subset of $\mathbb{R}^n$ containing $\mathbf{a}$, and $f : U \setminus \{\mathbf{a}\} \to \mathbb{R}^m$ a function. State what it means for $f(\mathbf{x})$ to converge to a vector $\mathbf{L} \in \mathbb{R}^m$ as $\mathbf{x} \to \mathbf{a}$.

(b) Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x^2 y}{x^2 + y^2}, x^2 + y^2 \right) = (0, 0).$