MATH 410.500
Spring 2014
Exam #2 Solutions

1. (a) true, (b) false, (c) false, (d) true, (e) true, (f) true, (g) false, (h) true, (i) false, (j) false.

2. (b) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a open cover of $A \cup B$. Then $\mathcal{U}$ is an open cover of both A and B, and so by the compactness assumption there exist finite sets $E, F \subseteq I$ such that $A \subseteq \bigcup_{i \in E} U_i$ and $B \subseteq \bigcup_{i \in F} U_i$. Then $\{U_i\}_{i \in E \cup F}$ is a finite subcollection of $\mathcal{U}$ such that $A \cup B \subseteq \bigcup_{i \in E \cup F} U_i$, which shows that $A \cup B$ is compact.

   (c) We can write $\mathbb{R}^2$ as the union of the closed balls $\overline{B_r(0)}$ centered at 0 with radii ranging over $r \in \mathbb{R}$. Since a closed ball is bounded, it is compact by the Heine-Borel theorem. However, $\mathbb{R}^2$ is not compact. For example, $\{B_r(0)\}_{r \in \mathbb{R}}$ is an open cover of $\mathbb{R}^2$ which does not have a finite subcover. Indeed if $\{B_{r_i}(0)\}_{i=1,\dots,n}$ is a finite subcollection of this cover then its union would not contain the point $(r, 0)$ where $r = 1 + \max\{r_1, \dots, r_n\}$.

3. (b) It is enough to check that each coordinate tends to 0 as $(x, y) \to (0, 0)$. Given an $\varepsilon > 0$, set $\delta = \varepsilon^2$. Then for all $(x, y) \in \mathbb{R}^2$ satisfying $\|(x, y)\| < \delta$ we have $x^2 + y^2 = \|(x, y)\|^2 < \sqrt{\delta} = \varepsilon$, and so $x^2 + y^2 \to 0$ as $(x, y) \to (0, 0)$. Next notice that for all $x, y \in \mathbb{R}$ we have $(x - y)^2 \geq 0$ and $(x + y)^2 \geq 0$ and hence $x^2 + y^2 \geq |2xy|$ by expanding. Therefore when $(x, y) \neq (0, 0)$ we have

$$\left| \frac{x^2 y}{x^2 + y^2} \right| \leq \left| \frac{x^2 y}{2xy} \right| \leq |x|.$$

   Now given an $\varepsilon > 0$ set $\delta = \varepsilon$. Then for all $(x, y) \neq (0, 0)$ satisfying $\|(x, y)\| < \delta$ we have

$$\left| \frac{x^2 y}{x^2 + y^2} \right| \leq |x| \leq \|(x, y)\| < \varepsilon,$$

   showing that $x^2 y / (x^2 + y^2) \to 0$ as $(x, y) \to (0, 0)$.