MATH 410.501
Final examination
May 5, 2014

Name: ______________________

ID#: ______________________

The exam consists of 6 questions. The point value for a question is written next to the question number. There is a total of 100 points. No aids are permitted.

1. [45] In each of the following 15 cases, indicate whether the given statement is true or false. No justification is necessary.

(a) Let $\mathbf{x}$ be a nonzero vector in $\mathbb{R}^2$. Then there are only finitely many vectors $\mathbf{y} \in \mathbb{R}^2$ such that $\mathbf{x} \cdot \mathbf{y} = 0$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function and set $A = \{(x, y) \in \mathbb{R}^2 : \cos y \leq x < y^2\}$. Then $f^{-1}(A)$ is an open set.

(c) The intersection of infinitely many compact subsets of $\mathbb{R}^n$ is compact.

(d) Let A be an $n \times n$ matrix and define the function $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(\mathbf{x}) = A\mathbf{x}$. Then f is differentiable at every point of $\mathbb{R}^n$.

(e) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function such that $|f(x, y)| \leq x^2$ for all $(x, y) \in \mathbb{R}^2$. Then f is differentiable at $(0, 0)$.

(f) Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

(g) Let $\{\mathbf{x}_k\}_{k=1}^{\infty}$ and $\{\mathbf{y}_k\}_{k=1}^{\infty}$ be bounded sequences in $\mathbb{R}^n$. Then there exist $1 \leq k_1 < k_2 < k_3 < \cdots$ such that the sequence $\{\mathbf{x}_{k_j} + \mathbf{y}_{k_j}\}_{j=1}^{\infty}$ converges.

(h) Let $\{f_n\}$ be a sequence of differentiable functions from $[0,1]$ to $\mathbb{R}$ which converges pointwise on $[0,1]$ to a function f. Then f is continuous.

(i) If a power series $\sum_{k=0}^{\infty} c_k x^k$ converges on a compact set $A \subseteq \mathbb{R}$, then it converges uniformly on A.

(j) If a power series $\sum_{k=0}^{\infty} c_k x^k$ converges on $\mathbb{R}$, then for every $\delta > 0$ there is a $k_0 \in \mathbb{N}$ such that $|c_k| \leq \delta^k$ for all $k \geq k_0$.

(k) The subset $\{(x,y) \in \mathbb{R}^2 : |x-y| = 1\}$ of $\mathbb{R}^2$ is connected.

(l) The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\mathbf{x}) = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^2$ is differentiable at the point $\mathbf{x} = (0, 0)$.

(m) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 function and let A be a bounded subset of $\mathbb{R}^n$. Then there exists an $M > 0$ such that $\|Df(\mathbf{x})\| \leq M$ for all $\mathbf{x} \in A$.

(n) Let A be a bounded subset of $\mathbb{R}^n$. Then the closure $\overline{A}$ is compact.

(o) $\mathbb{R}^n$ can be written as the union of finitely many compact sets.

2. [15] (a) Let A be a nonempty subset of $\mathbb{R}$. State what it means for a sequence $\{f_n\}$ of functions to converge uniformly on A to a function f.

(b) State what it means for a set $A \subseteq \mathbb{R}^n$ to be compact.

(c) Define the interior A° and closure $\overline{A}$ of a set $A \subseteq \mathbb{R}^n$.

3. [10] (a) Compute all second-order partial derivatives of the function $f(x, y) = \sin(e^{xy})$.

(b) Is the function in part (a) differentiable at $(0, 0)$? (No justification is required.)

4. [10] (a) State what it means for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable at a point $\mathbf{x} \in \mathbb{R}^n$.

(b) Prove directly from the definition of differentiability that the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 + z^2$ is differentiable at $(0, 0, 0)$.

(c) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the first-order partial derivatives of f exist at $(0, 0)$ but f is not differentiable at $(0, 0)$. No justification is required.

5. [10] Determine Taylor's formula for the function $f(x, y) = \ln(x + 2y)$ at the point $(1, 0)$ with remainder term of order three (i.e., the remainder term contains the third total differential).

6. [10] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (e^{xy}, x + y)$. Prove that f is invertible on some open set U containing $(1, 0)$ and that the inverse f^{-1} is differentiable on $f(U)$, and compute $D(f^{-1})(1, 1)$.