MATH 410.500
Spring 2014
Final exam solutions

1. (a) false, (b) false, (c) true, (d) true, (e) true, (f) true, (g) true, (h) false, (i) true, (j) true, (k) false, (l) false, (m) true, (n) true, (o) false.

3. (b)

$$\frac{\partial f}{\partial x} = ye^{xy}\cos(e^{xy}), \quad \frac{\partial f}{\partial y} = xe^{xy}\cos(e^{xy})$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy}(\cos(e^{xy}) - e^{xy}\sin(e^{xy}))$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = e^{xy}((1+xy)\cos(e^{xy}) - xye^{xy}\sin(e^{xy}))$$

$$\frac{\partial^2 f}{\partial y^2} = e^{xy}(-x^2 e^{xy}\sin(e^{xy}) + xy\cos(e^{xy})).$$

(d) Yes (since it has continous first-order partial derivatives).

4. (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T\mathbf{h} = 0$ for all $\mathbf{h} \in \mathbb{R}^2$. Then for $\mathbf{h} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ we have

$$\frac{|f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - T\mathbf{h}|}{\|\mathbf{h}\|} = \frac{\|\mathbf{h}\|^2}{\|\mathbf{h}\|} = \|\mathbf{h}\|,$$

and this last quantity tends to zero as $\|\mathbf{h}\| \to 0$. Therefore f is differentiable at $\mathbf{0}$.

(c) $f(x,y) = 0$ if $xy = 0$ and $f(x,y) = 1$ otherwise.

5.

$$f(x,y) = x - 1 + 2y + \frac{1}{2}\big(-(x-1)^2 - 4(x-1)y - 4y^2\big) + \frac{1}{6}\cdot\frac{1}{(c_1+2c_2)^3}\big(2(x-1)^3 + 12(x-1)^2 y - 24(x-1)^2 y^2 + 16y^3\big)$$

where (c_1, c_2) is some point on the line joining $(1,0)$ and (x,y).

6. Writing f_1 and f_2 for the components of f, we have $\partial f_1/\partial x = ye^{xy}$, $\partial f_1/\partial y = xe^{xy}$, $\partial f_2/\partial x = 1$, and $\partial f_2/\partial y = 1$. Thus

$$Df(x,y) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and hence $\Delta_f(1,0) = 1 \neq 0$. It also follows that f is C^1 on $\mathbb{R}^2$, and so by the inverse function theorem there is an open set $U \subseteq \mathbb{R}^2$ containing $(1,0)$ such that

(i) f is injective on U, and

(ii) f^{-1} is differentiable on $f(U)$ and $Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$ for all $y \in U$.

Therefore

$$Df^{-1}(1,1) = [Df(1,0)]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$