MATH 608 Final Exam
Spring 2011

Instructions: Do one problem from each of the following sets: $\{1\}$, $\{2\}$, $\{3,4\}$, $\{5,6\}$.

1. (a) Show that, in a normed vector space, the closure of a subspace is a subspace.

(b) Let X be a Banach space and V be a proper closed subspace of X. Consider X/V with the quotient norm $\|x+V\| = \inf_{y\in V}\|x+y\|$. Show that the projection map $\pi(x) = x+V$ from X to X/V has norm 1.

(c) Give a direct proof (i.e., without using the open mapping theorem) that the projection map π in part (b) is open.

2. Let $1 < p < \infty$, and let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\ell^p(\mathbb{N})$ such that $\lim_{k\to\infty} x_k(n) = 0$ for all $n \in \mathbb{N}$. Show that if there is an $M > 0$ such that $\|x_k\| \leq M$ for all $k \in \mathbb{N}$ then $x_k \to 0$ weakly. Also, show that if there is no such M then $\{x_k\}_{k=1}^{\infty}$ can fail to converge weakly.

3. (a) State what it means for a topological space X to be compact.

(b) Let (X,d) be a compact metric space. Recall that the *diameter* of a set $Y \subseteq X$ is the number $\sup\{d(x,y) : x,y \in Y\}$, that the *support* of a Borel measure μ on X is the complement of the union of all open sets $U \subseteq X$ such that $\mu(U) = 0$, and that a *point mass* is a Borel measure δ_x which corresponds to some $x \in X$ and is given by $\delta_x(A) = 1$ or 0 depending on whether or not $x \in A$.

Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Borel probability measures on X which converges in the weak* topology to a Borel measure μ. Show that if the diameter of the support of μ_n tends to zero as $n \to \infty$ then μ is a point mass. Also, show that the converse is false in general.

4. (a) State the complex Hahn-Banach theorem.

(b) Let X be a separable Banach space. Show that there exists an isometric linear map from X into $\ell^\infty(\mathbb{N})$. Also, show that this is false in general if $\ell^\infty(\mathbb{N})$ is replaced by $\ell^2(\mathbb{N})$.

5. (a) State the Riesz representation theorem for $C_0(X)^*$ where X is a locally compact Hausdorff space.

(b) Let X be a compact Hausdorff space which has a basis of clopen sets. Let μ be a Borel probability measure on X and $\{\mu_i\}$ a net of Borel probability measures on X. Prove that $\mu_i \to \mu$ in the weak* topology if and only if $\mu_i(A) \to \mu(A)$ for every clopen set $A \subseteq X$.

6. (a) State Tychonoff's theorem.

(b) For each $n \in \mathbb{N}$ let $0 \leq a_n < b_n \leq 1$. Set $Y = \prod_{n\in\mathbb{N}}(a_n, b_n)$, viewed as a subset of $X = [0,1]^{\mathbb{N}}$ in the obvious way. In each of the following cases, give conditions on the a_n and b_n that are equivalent to the stated property.

(1) Y is open in X.
(2) Y is dense in X.
(3) Y is nowhere dense in X (i.e., $\overline{Y}$ has empty interior).

Justify your answers.