Midterm Exam
MATH 608
Spring 2011

Instructions: Do 4 out of the 6 problems, including #1, #2, one of #3 and #4, and one of #5 and #6.

1. (a) Let X be a Hausdorff space. Show that every net in X has at most one limit.

(b) Give an example, with explanation, of a net in a topological space which converges to more than one point.

2. (a) State the open mapping theorem.

(b) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X. Suppose that X is complete with respect to both norms, and suppose that there is a $C > 0$ such that $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$. Show that the two norms are equivalent.

3. (a) State what it means for a topological space X to be compact.

(b) Let X be a compact space and let $g : X \to \mathbb{C}$ be a continuous function such that $g(x) \neq 0$ for all $x \in X$. Show that the function $h : X \to \mathbb{C}$ defined by $h(x) = 1/g(x)$ for all $x \in X$ is continuous.

(c) Let X be a compact space and let $\mathcal{I}$ be an ideal in $C(X)$ such that for every $x \in X$ there is an $f \in \mathcal{I}$ satisfying $f(x) \neq 0$. Show that $\mathcal{I} = C(X)$.

4. (a) State the complex Stone-Weierstrass theorem.

(b) Let X and Y be compact Hausdorff spaces, and let $\mathcal{A}$ be a proper closed subalgebra of $C(X)$ which is closed under complex conjugation. Show that the algebra generated by the functions of the form $f(x, y) = g(x)h(y)$ for $g \in \mathcal{A}$ and $h \in C(Y)$ is not dense in $C(X \times Y)$.

5. (a) State the complex Hahn-Banach theorem.

(b) Let X be a Banach space and V a finite-dimensional subspace of X. Let $\varepsilon > 0$. Show that there is an $n \in \mathbb{N}$ and a bounded linear map $T : X \to \mathbb{C}^n$ such that $\|T\| = 1$ and $\|Tx\| \geq (1-\varepsilon)\|x\|$ for all $x \in V$, where $\mathbb{C}^n$ is equipped with the norm $\|(z_1, \ldots, z_n)\| = \max_{i=1,\ldots,n} |z_i|$.

6. (a) State Urysohn's lemma.

(b) Let X be a compact Hausdorff space. Suppose that the set of continuous complex-valued functions on X with finite image is dense in $C(X)$ with respect to the uniform norm. Show that the topology on X has a base consisting of sets which are both open and closed.

Solutions

1. (a) Given two distinct points $x, y \in X$ we can find, by hypothesis, disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. Then any net in X cannot be both eventually in U and eventually in V. Consequently every net in X has at most one limit.

(b) Consider a two-point space $X = \{x_1, x_2\}$ with the coarse topology. Then every net in X converges to both x_1 and x_2.

2. (b) Consider the identity map $\iota : X \to X$ where the first copy of X is equipped with $\|\cdot\|_2$ and the second copy with $\|\cdot\|_1$. By hypothesis, ι is continuous. Since it is bijective, by the open mapping theorem ι^{-1} is continuous, and so there is a $D > 0$ such that $\|x\|_2 \leq D\|x\|_1$ for all $x \in X$, showing that the two norms are equivalent.

3. (b) The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = z^{-1}$ is readily seen to be continuous from the observation that $|z^{-1} - w^{-1}| \leq |zw|^{-1}|z - w|$ for all $z, w \in \mathbb{C} \setminus \{0\}$. Therefore the composition $h = f \circ g = 1/g$ is continuous.

(c) For every $x \in X$ take an $f_x \in \mathcal{I}$ such that $f(x) \neq 0$, and then an open neighborhood U_x of x such that $f(y) \neq 0$ for all $y \in U_x$, which exists by the continuity of f_x. Since $\{U_x\}_{x \in X}$ is an open cover of X it admits by compactness a finite subcover $\{U_x\}_{x \in F}$. Set $g = \sum_{x \in F} \bar{f}_x f_x$, which lies in $\mathcal{I}$ since $\mathcal{I}$ is an ideal. Then $g(x) \neq 0$ for all $x \in X$ and so $1/g$ exists and is continuous by part (a). Since $\mathcal{I}$ is an ideal we have $1 = g \cdot (1/g) \in \mathcal{I}$ and hence $f = f \cdot 1 \in \mathcal{I}$ for all $f \in C(X)$.

4. (b) Write $\mathcal{B}$ for the subalgebra of $C(X \times Y)$ generated by the functions of the form $f(x, y) = g(x)h(y)$ for $g \in \mathcal{A}$ and $h \in C(Y)$. Suppose first that $\mathcal{A}$ separates points. Since by hypothesis $\mathcal{A}$ is not dense in $C(X)$, by the Stone-Weierstrass theorem we have $\mathcal{A} = \{g \in C(X) : g(x_0) = 0\}$ for some $x_0 \in X$. It follows that $f(x_0, y) = 0$ for all $f \in \mathcal{B}$ and $y \in Y$, and so the non-zero constant functions do not lie in the closure of $\mathcal{B}$, whence $\mathcal{B}$ is not dense.

Suppose now that $\mathcal{A}$ does not separate points. Then there are distinct $x_1, x_2 \in X$ such that $g(x_1) = g(x_2)$ for all $g \in \mathcal{A}$. Pick a $y \in Y$. Then $f(x_1, y) = f(x_2, y)$ for all $f \in \mathcal{B}$. By Urysohn's lemma there is an $h \in C(X \times Y)$ such that $h((x_1, y)) = 0$ and $h((x_2, y)) = 1$. Then $\|f - h\| \geq 1/2$ for all $f \in \mathcal{B}$, so that $\mathcal{B}$ is not dense.

5. (b) Since V is finite-dimensional its closed unit ball B is compact, and so we can find finite set $\{x_1, \dots, x_n\} \subseteq B$ which is $(\varepsilon/2)$-dense in B. For each $i = 1, \dots, n$ we can find, by the Hahn-Banach theorem, a bounded linear functional $\varphi_i : X \to \mathbb{C}$ of norm one such that $|\varphi_i(x_i)| = \|x_i\|$. Define the linear map $\psi : X \to \mathbb{C}^n$ by $\psi(x) = (\varphi_1(x), \dots, \varphi_n(x))$. Then ψ has norm one by the definition of the norm on $\mathbb{C}^n$. Now if x is an element of V of norm one then there is an $i \in \{1, \dots, n\}$ such that $\|x - x_i\| \leq \varepsilon/2$, so that

$$|\varphi_i(x)| \geq |\varphi_i(x_i)| - |\varphi_i(x_i - x)| \geq \|x_i\| - \|x_i - x\| \geq \|x\| - 2\|x_i - x\| \geq 1 - \varepsilon.$$

and hence $|\psi(x)| \geq 1 - \varepsilon$. Thus by scaling $|\psi(x)| \geq (1-\varepsilon)\|x\|$ for all $x \in V$.

6. (b) Let U be a nonempty open subset of X and let $x \in U$. It suffices to show that there is a clopen set V such that $V \subseteq U$ and $x \in V$. By Urysohn's lemma there is an $f \in C(X)$ such that $f = 0$ on $X \setminus U$ and $f(x) = 1$. By hypothesis there is a $g \in C(X)$ with finite image such that $\|f - g\|_u < 1/4$. Since g is continuous and has finite image, the set $V = g^{-1}(\{z \in \mathbb{C} : |z| > 1/2\})$ is both closed and open, and $|g(y)| \leq 1/4$ for all $y \in X \setminus U$ while $|g(x)| \geq 3/4$, so that V satisfies the desired properties.