Midterm Exam
MATH 608
Spring 2019

Instructions: Do 4 out of the 7 problems, including at least one from the set $\{5, 6, 7\}$.

1. (a) Let X be a topological space, Y a Hausdorff space, and $f, g : X \to Y$ continuous maps. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed.

(b) Give an example to show that part (a) fails if Y is not assumed to be Hausdorff.

2. (a) Let X be a normed vector space and let V be a proper closed subspace of X. Show that $\|x + V\| = \inf\{\|x + y\| : y \in V\}$ defines a norm on X/V.

(b) Show that the projection $\pi : X \to X/V$ defined by $\pi(x) = x + V$ has norm one.

3. (a) State the complex Stone-Weierstrass theorem.

(b) Let X and Y be compact Hausdorff spaces. Show that the algebra generated by the functions of the form $f(x, y) = g(x)h(y)$ for $g \in C(X)$ and $h \in C(Y)$ is a dense subset of $C(X \times Y)$.

4. (a) State the open mapping theorem.

(b) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X. Suppose that X is complete with respect to both norms and that $\|x\|_1 \leq \|x\|_2$ for all $x \in X$. Show that the two norms are equivalent.

(c) Show that a nonsurjective linear map between Banach spaces is never open.

5. (a) State what it means for a topological space X to be compact.

(b) Let (X, d) be a compact metric space and let U be an open subset of $X \times X$ containing the diagonal $\{(x, x) : x \in X\}$. Show that there exists an $\varepsilon > 0$ such that $\{(x, y) \in X \times X : d(x, y) < \varepsilon\} \subseteq U$.

(c) Give an example to show that (b) fails if X is not assumed to be compact.

6. (a) State the complex Hahn-Banach theorem.

(b) Let X be a Banach space and V a finite-dimensional subspace of X. Let $\varepsilon > 0$. Using the fact that the unit ball of V is compact, show that there is a bounded linear map $T : X \to C([0, 1])$ such that $(1 - \varepsilon)\|x\| \leq \|Tx\| \leq \|x\|$ for all $x \in V$.

7. (a) State Urysohn's lemma.

(b) Let $n \in \mathbb{N}$ and equip $\mathbb{C}^n$ with the norm $\|(x_i)\| = \max_i |x_i|$. Let Z be an infinite compact Hausdorff space. Show that there exists an isometric linear map $T : \mathbb{C}^n \to C(Z)$.

Solutions

4. (c) Let $T : X \to Y$ be an open linear map between Banach spaces. Then there is an $\varepsilon > 0$ such that the image under T of the open ball $B(1, 0) \subseteq X$ contains the open ball $B(\varepsilon, 0) \subseteq Y$. But then for every $n \in \mathbb{N}$ we have $T(B(n, 0)) = nT(B(1, 0)) \supseteq nB(\varepsilon, 0) = B(n\varepsilon, 0)$, and since $Y = \bigcup_{n=1}^{\infty} B(n\varepsilon, 0)$ we conclude that T is surjective.

5. (b) We equip $X \times X$ with the product metric. For every $x \in X$ take an $\varepsilon_x > 0$ such that the ball $B(\varepsilon_x, (x, x))$ of radius ε_x around (x, x) is contained in U. Sinxe X is compact so is $X \times X$, and so the diagonal in $X \times X$, being a closed subset of $X \times X$ by Hausdorffness, is compact. We can thus find $x_1, \dots, x_n$ such that the balls $B(\varepsilon_{x_i}/2, (x_i, x_i))$ for $i = 1, \dots, n$ cover $X \times X$. Set ε to be the minimum of $\varepsilon_{x_1}/2, \dots, \varepsilon_{x_n}/2$. Now let $x, y \in X$ be such that $d(x, y) < \varepsilon$. Take an $1 \leq i \leq n$ such that $(x, x) \in B(\varepsilon_{x_i}/2, (x_i, x_i))$. Then, writing ρ for the product metric on $X \times X$,

$$\rho((x, y), (x_i, x_i)) \leq \rho((x, y), (x, x)) + \rho((x, x), (x_i, x_i)) < \varepsilon + \frac{\varepsilon_{x_i}}{2} \leq \varepsilon_{x_i}$$

which shows that (x, y) belongs to the ball $B(\varepsilon_{x_i}, (x_i, x_i))$ and hence to U.

(c) Take $X = \mathbb{R}$ and take U to be the set $\{(x, y) \in \mathbb{R}^2 : x - e^{-x^2} < y < x + e^{-x^2}\}$.

6. (a) Since V is finite-dimensional its closed unit ball B is compact, and so we can find finite set $\{x_1, \dots, x_n\} \subseteq B$ which is $(\varepsilon/2)$-dense in B. For each $i = 1, \dots, n$ we can find, by the Hahn-Banach theorem, a bounded linear functional $\omega_i : X \to \mathbb{C}$ of norm one such that $|\omega_i(x_i)| = \|x_i\|$. Define the linear map $\omega : X \to \mathbb{C}^n$ by $\psi(x) = (\omega_1(x), \dots, \omega_n(x))$. Then φ has norm one by the definition of the norm on $\mathbb{C}^n$. Now if x is an element of V of norm one then there is an $i \in \{1, \dots, n\}$ such that $\|x - x_i\| \leq \varepsilon/2$, so that

$$|\omega_i(x)| \geq |\omega_i(x_i)| - |\omega_i(x_i - x)| \geq \|x_i\| - \|x_i - x\| \geq \|x\| - 2\|x_i - x\| \geq 1 - \varepsilon.$$

and hence $|\varphi(x)| \geq 1-\varepsilon$. Thus by scaling $|\varphi(x)| \geq (1-\varepsilon)\|x\|$ for all $x \in V$. Next take continuous functions (for example, tent functions) $h_1, \dots, h_n : [0, 1] \to [0, 1]$ with pairwise disjoint supports such that $\|h_i\| = 1$ for each $i = 1 \dots, n$, and define the linear map $\psi : \mathbb{C}^n \to C([0, 1])$ by $\psi((z_i)_{i=1}^n) = \sum_{i=1}^n z_i h_i$. Then ψ is isometric, and so we may take T to be $\psi \circ \varphi$.

7. (a) Since X is infinite we can find n distinct points $x_1, \dots, x_n \in X$. Since X is Hausdorff, for all $1 < i, j < n$ with $i \neq j$ we can find disjoint open sets $U_{i,j}, V_{i,j} \subseteq X$ such that $x_i \in U_{i,j}$ and $x_j \in V_{i,j}$. For each $i = 1, \dots, n$ set $U_i = \bigcap_{j \neq i}(U_{i,j} \cap V_{j,i})$. Then the sets $U_1, \dots, U_n$ are open and pairwise disjoint and $x_i \in U_i$ for every i. By Urysohn's lemma, for each $i = 1, \dots, n$ we can find a continuous function $h_i : Z \to [0, 1]$ such that $x_i \in h_i$ and $h_i(x) = 0$ for all $x \in Z \setminus U_i$. Now define the linear map $T : \mathbb{C}^n \to C(Z)$ by $T((z_i)_{i=1}^n) = \sum_{i=1}^n z_i h_i$, which is clearly isometric.