Home Assignment # 1, M423, Spring 2017
Due January 24th
1 point per a problem

Section 500 only

1. Prove that if one of the vectors in a set $x_1, \ldots, x_m$ is a linear combination of others, then the vectors are linearly dependent

2. Prove that if vectors $x_1, \ldots, x_m$ are linearly independent, then any subset of them are also linearly independent.

3. Prove that any n-dimensional vector space has subspaces of all dimensions $j = 0, 1, \ldots, n$.

4. Compute the volume of the parallelepiped in $\mathbb{R}^3$ generated by vectors $(1, -1, 0), (2, 1, 2)$, and $(2, 2, 1)$.

5. Let P be the space of polinomials in variable t of degree up to 5 and D is the transformation of taking the second derivative: $D(p(t)) := p''(t)$. Find the matrix of D in the basis $1, t, t^2, t^3, t^4, t^5$.

Sections 500 and 200

6. How does operator D^2acts on a polynomial and what is its matrix?

7. The same about D^3.

8. Prove that any set of vectors containing the zero vector is linearly dependent.

9. Prove that the vector space $M_{m,n}$ of $m \times n$-matrices has dimension mn.

10. Show that if some collection of vectors is linearly dependent, then adding extra vectors does not make it linearly independent.

11. If a collection of vectors is linearly independent, can adding extra vectors make it dependent? If no, prove, if yes, give an example.

12. Prove that the dimension of a linear subspace W of a vector space V cannot exceed the dimension of V.

13. Prove that in the vector space of n-tuples $(\xi_1, \dots, \xi_n)$ the set of vectors satisfying

$$\alpha_1\xi_1 + \cdots + \alpha_n\xi_n = 0$$

for a given set a scalars $(\alpha_1, ..., \alpha_n)$, **not all of which are equal to zero**, forms a subspace of dimension $n - 1$.

14. Prove that if the two subspaces V_1, V_2 have only zero in common, then the sum of their dimensions cannot exceed the dimension of the whole space V.

Section 200 only

15. Prove that if the dimension of a linear subspace W of a vector space V is equal to the dimension of V, then $W = V$.

16. Prove that if the set of vectors $x_1, \dots, x_k$, $k < n$, in an n-dimensional vector space can always be extended to a basis $x_1, \dots, x_n$.

17. Prove that if the two subspaces V_1, V_2 have only zero in common and if the sum of their dimensions is equal the dimension of the whole space V, then any vector $x \in V$ has unique representation $x = y_1 + y_2$, where $y_j \in V_j$, $j = 1, 2$.

18. Show that the dimension of the subspace generated by a set of vectors $x_1, \dots, x_k$ is equal the maximal number of linearly independent vectors among them.