Let

$$F(x) + C = \int f(x)dx \qquad f(x) + C = \int f'(x)dx \qquad g(x) + C = \int g'(x)dx$$

Here $F(x) + C$ is an antiderivative for $f(x)$, and $f(x) + C$ is an antiderivative for $f'(x)$.

Integration by substitution

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$$

$u = g(x),\ du = g'(x)dx$.

Example:

$$\int x\sin(x^2+1)dx$$

The difficulty here is that we don't know how to integrate $x\sin(x^2+1)$, but notice that the derivative of x^2+1 is just $2x$, and we see the x part of $2x$ multiplying the $\sin(x^2+1)$.

Let $u = x^2+1$ then $du = 2xdx$ or $xdx = (1/2)du$.

$$\int x\sin(x^2+1)dx = \frac{1}{2}\int \sin(u)du = -\cos(u) + C = -\cos(x^2+1) + C$$

Integration by parts

This comes from the product rule. What is the derivative of uv?

$$[uv]' = u'v + uv'$$
$$\int [uv]' = uv = \int u'vdx + \int uv'dx \qquad \text{(here we ignore the constant of integration)}$$

Let $v'dx = dv$ and $u'dx = du$ and rearrange the above to solve for $\int udv$ and you get:

$$\int udv = uv - \int vdu$$

In doing integration by parts we always choose u to be something we can differentiate, and dv to be something we can integrate. A useful rule for figuring out what to make u is the LIPET rule. Choose u as the first function that appears on the following list: Logarithms, Inverse trig, Polynomials, Exponentials, Trigonometry.

If we need to include limits of integration

$$\int_a^b udv = uv\Big]_a^b - \int_a^b vdu$$

Example:

$$\int x\sin xdx$$

Let $u = x$ and $dv = \sin xdx$. Then $du = dx$ and $v = -\cos x$. We get

$$\int x\sin xdx = -x\cos x + \int \cos xdx$$
$$= -x\cos x + \sin x + C$$

Integration by partial fractions

We'll look at this by example. Consider the integral

$$\int \frac{x-1}{x^2+x}dx$$

We write the integrand as the sum of two fractions that are *simpler*. The denominator is $x(x+1)$, and we attempt

$$\frac{x-1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$$

I need to solve for A and B. Find a common denominator

$$\frac{x-1}{x^2+x} = \frac{A(x+1)}{x^2+x} + \frac{Bx}{x^2+x}$$

and set like terms in the numerator to be equal. The term with x is $x = Ax + Bx$ so $A + B = 1$. The constant term is $-1 = A$. The solution is $A = -1$ and $B = 2$. Putting this back in the original integral:

$$\int \frac{x-1}{x^2+x}dx = \int \left(\frac{-1}{x} + \frac{2}{x+1}\right) dx = -\ln(|x|) + 2\ln(|x+1|) + C = \ln\left(\frac{(x+1)^2}{|x|}\right) + C$$

We do not need the absolute values on $(x+1)^2$ because a squared number is never negative.

Practice

Now it's your turn. Some of these integrals are pretty easy and you should have them in memory. Some are harder and use the techniques we just went over.

1. $\int x^2 dx =$
   this is an indefinite integral, an antiderivative

2. $\int x^{-3} dx =$

3. $\int (1/x) dx =$

4. $\int_{-3}^{3} \sin(x) dx =$
   this is a definite, not an indefinite, integral

5. $\int_0^{\pi/2} \cos(2x)dx =$

6. $\int \tan(x)dx =$

$\tan(x) = \sin(x)/\cos(x);$ substitution

7. $\int (x^2 + 1/x^2)dx =$

8. $\int_{-\infty}^{0} 6e^{2x}dx =$

9. $\int x\exp(x^2)dx =$

10. $\int (2x+3)\sin(x^2+3x)dx =$

11. $\int xe^x dx =$

12. $\int \ln(x)dx =$

by parts

13. $\int \frac{1}{(x^2-1)}dx =$

14. $\int (\ln(x)/x)\,dx =$