Lecture I: Asymptotics for large GUE random matrices

Steen Thorbjørnsen, University of Aarhus

Random Matrices

Definition. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let n be a positive integer. Then a random $n \times n$ matrix A on $(\Omega, \mathcal{F}, P)$ is an $n \times n$-matrix

$$A = (a_{ij})_{1 \le i,j \le n},$$

where all the entries are complex valued random variables on $(\Omega, \mathcal{F}, P)$. In other words, A is a measurable mapping

$$A \colon (\Omega, \mathcal{F}, P) \to (M_n(\mathbb{C}), \mathcal{B}(M_n(\mathbb{C}))),$$

when $M_n(\mathbb{C})$ is equipped with its Borel σ-algebra $\mathcal{B}(M_n(\mathbb{C}))$.

The spectral distribution of a selfadjoint random matrix

Let $A\colon \Omega \to M_n(\mathbb{C})$ be a selfadjoint random matrix, i.e., $A^*(\omega) = A(\omega)$ for all ω. Then for each ω we consider the ordered eigenvalues

$$\lambda_1(\omega) \leq \lambda_2(\omega) \leq \cdots \leq \lambda_n(\omega)$$

of $A(\omega)$. For each fixed ω the *empirical eigenvalue distribution of* $A(\omega)$ is the probability measure

$$\mu_{A(\omega)} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j(\omega)},$$

where δ_c denotes the Dirac measure at the constant c.
Then the *spectral distribution* of A is the *mixture* of the family $(\mu_{A(\omega)})_{\omega\in\Omega}$ with respect to P, i.e.......

The spectral distribution (continued)

$$\mu_A(B) = \int_\Omega \mu_{A(\omega)}(B)\,\mathrm{d}P(\omega),$$

for any Borel subset B of $\mathbb{R}$. Thus, by a standard extension argument, for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned}\int_{\mathbb{R}} f(t)\,\mu_A(\mathrm{d}t) &= \int_\Omega \left(\int_{\mathbb{R}} f(t)\,\mu_{A(\omega)}(\mathrm{d}t) \right) \mathrm{d}P(\omega) \\ &= \int_\Omega \frac{1}{n}\sum_{j=1}^{n} f(\lambda_j(\omega))\,\mathrm{d}P(\omega) \\ &= \int_\Omega \mathrm{tr}_n\big(f(A(\omega))\big)\,\mathrm{d}P(\omega) \\ &= \mathbb{E}\big\{\mathrm{tr}_n\big(f(A)\big)\big\}.\end{aligned}$$

The Gaussian Unitary Ensemble

Definition. By $\mathrm{GUE}(n, \sigma^2)$ we denote the set of random $n \times n$ matrices $X = (x_{ij})_{1 \le i,j \le n}$, defined on $(\Omega, \mathcal{F}, P)$, which satisfy the following conditions:

- $\forall i \geq j \colon x_{ij} = \overline{x_{ji}}$.
- the random variables x_{ij}, $1 \le i \le j \le n$, are independent.
- $\forall i < j \colon \mathrm{Re}(x_{ij}), \mathrm{Im}(x_{ij}) \sim \text{i.i.d. } N(0, \frac{1}{2}\sigma^2)$.
- $\forall i \colon x_{ii} \sim N(0, \sigma^2)$.

The spectral distribution of a GUE random matrix.

Let X_n be a random matrix in $\mathrm{GUE}(n, \frac{1}{n})$. For any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$ we then have

$$\mathbb{E}\{\mathrm{tr}_n(f(X_n))\} = \int_{\mathbb{R}} f(x) h_n(x)\,\mathrm{d}x,$$

where the function $h_n\colon \mathbb{R} \to \mathbb{R}$ is given by

$$h_n(x) = \frac{1}{\sqrt{2n}} \sum_{j=0}^{n-1} \varphi_j\big(\sqrt{\tfrac{n}{2}}x\big)^2,$$

and where

- $\varphi_0, \varphi_1, \varphi_2, \ldots,$ is the sequence of Hermite functions:

$$\varphi_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} H_k(x) \exp(-\tfrac{x^2}{2}), \qquad (k \in \mathbb{N}_0),$$

- $H_0, H_1, H_2, \ldots,$ are the Hermite polynomials:

$$H_k(x) = (-1)^k \exp(x^2) \cdot \Big(\frac{d^k}{dx^k} \exp(-x^2)\Big).$$

The moment generating function of a GUE random matrix

Theorem. Let X_n be a random matrix in $\mathrm{GUE}(n, \frac{1}{n})$. Then for any complex number z we have

$$\mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} = \exp(\tfrac{z^2}{2n}) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \left(\frac{z^2}{n}\right)^j.$$

Sketch of Proof.

By analytic continuation it suffices to consider the case $z \in \mathbb{R}$.
Step I. We know that

$$\mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} = \frac{1}{\sqrt{2n}} \int_{\mathbb{R}} \exp(zt)\Big(\sum_{k=0}^{n-1} \varphi_k\big(\sqrt{\tfrac{n}{2}}t\big)^2\Big)\,\mathrm{d}t,$$

so by a substitution, it follows that we have to show that

$$F(s) := \int_{\mathbb{R}} \exp(st)\Big(\sum_{k=0}^{n-1} \varphi_k(t)^2\Big)\,\mathrm{d}t$$

$$= n\exp(\tfrac{s^2}{4}) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\Big(\frac{s^2}{2}\Big)^j,$$

for any real number s.

Sketch of Proof (continued).

Step II. By partial integration we find that

$$F(s) := \int_{\mathbb{R}} \exp(st)\Big(\sum_{k=0}^{n-1} \varphi_k(t)^2\Big)\,\mathrm{d}t = -\frac{1}{s}\int_{\mathbb{R}} \exp(st)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{k=0}^{n-1} \varphi_k(t)^2\Big)\,\mathrm{d}t$$

and using properties of the Hermite polynomials, one may verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{k=0}^{n-1} \varphi_k(t)^2\Big) = -\sqrt{2n}\varphi_n(t)\varphi_{n-1}(t),$$

so that

$$F(s) = \frac{\sqrt{2n}}{s}\int_{\mathbb{R}} \exp(st)\varphi_n(t)\varphi_{n-1}(t)\,\mathrm{d}t$$

$$= \frac{\sqrt{2n}}{s}\big(2^{2n-1}n!(n-1)!\pi\big)^{-1/2}\int_{\mathbb{R}} \exp(st-t^2)H_n(t)H_{n-1}(t)\,\mathrm{d}t.$$

Sketch of Proof (continued).

Step III Using the substitution $y = t - \frac{1}{2}s$ we have that $\exp(st - t^2) = \exp(-y^2)\exp(\frac{1}{4}s^2)$, and hence

$$F(s) = \frac{\sqrt{2n}}{s}\left(2^{2n-1}n!(n-1)!\pi\right)^{-1/2}\int_{\mathbb{R}}\exp(st - t^2)H_n(t)H_{n-1}(t)\,\mathrm{d}t$$

$$= \frac{1}{s}\left(2^{(n-1)}((n-1)!)^2\pi\right)^{-1/2}\exp(\tfrac{1}{4}s^2) \times \int_{\mathbb{R}}\exp(-y^2)H_n(y+\tfrac{1}{2}s)H_{n-1}(y+\tfrac{1}{2}s)\,\mathrm{d}y.$$

Using then the following properties of the Hermite polynomials:

$$H_k(x+a) = \sum_{j=0}^{k}\binom{k}{j}(2a)^{k-j}H_j(x), \qquad (x, a \in \mathbb{R},\ k \in \mathbb{N})$$

and

$$\int_{\mathbb{R}} H_k(y)H_l(y)\exp(-y^2)\,\mathrm{d}y = \delta_{k,l}(2^k k!\sqrt{\pi}) \qquad (k, l \in \mathbb{N}_0),$$

Sketch of Proof (continued).

we find that

$$F(s) = \frac{1}{s}\left(2^{(n-1)}((n-1)!)^2\pi\right)^{-1/2}\exp(\tfrac{1}{4}s^2)$$
$$\times \sum_{j=0}^{n-1}\binom{n}{j}\binom{n-1}{j}s^{n-j}s^{n-1-j}(2^j j!\sqrt{\pi})$$

$$= \cdots$$

$$= n\exp(\tfrac{s^2}{4})\sum_{j=0}^{n-1}\frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\left(\frac{s^2}{2}\right)^j,$$

as desired. ■

Wigner's semi-circle law

Theorem. For each n in $\mathbb{N}$, let X_n be a random matrix in $\mathrm{GUE}(n, \frac{1}{n})$ and consider its spectral distribution μ_{X_n}. Then

$$\mu_{X_n} \xrightarrow{\mathrm{w}} \tfrac{1}{2\pi}\sqrt{4-t^2}1_{[-2,2]}(t)\,\mathrm{d}t, \qquad \text{as } n \to \infty,$$

i.e., for any continuous bounded function $f\colon \mathbb{R} \to \mathbb{R}$ we have

$$\mathbb{E}\{\mathrm{tr}_n(f(X_n))\} = \int_{\mathbb{R}} f(x)\,\mu_{X_n}(\mathrm{d}x) \longrightarrow \frac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}\,\mathrm{d}x,$$

as $n \to \infty$.

Proof of Wigner's semi-circle law.

By the continuity theorem for characteristic functions of probability measures, it suffices to show that

$$\mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} \longrightarrow \frac{1}{2\pi}\int_{-2}^{2} \exp(zt)\sqrt{4-t^2}\,\mathrm{d}t, \qquad \text{as } n \to \infty$$

for any complex number z. Given such a z it follows by the previous theorem and dominated convergence (for series) that

$$\mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} = \exp(\tfrac{z^2}{2n}) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\Big(\frac{z^2}{n}\Big)^j$$

$$\underset{n\to\infty}{\longrightarrow} \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} z^{2j}.$$

Proof of Wigner's semi-circle law (continued).

On the other hand we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-2}^{2}\exp(zt)\sqrt{4-t^2}\,\mathrm{d}t &= \frac{1}{2\pi}\int_{-2}^{2}\Big(\sum_{j=0}^{\infty}\frac{z^j t^j}{j!}\Big)\sqrt{4-t^2}\,\mathrm{d}t \\
&= \sum_{j=0}^{\infty}\frac{z^j}{j!}\Big(\frac{1}{2\pi}\int_{-2}^{2}t^j\sqrt{4-t^2}\,\mathrm{d}t\Big) \\
&= \sum_{j=0}^{\infty}\frac{z^{2j}}{(2j)!}\Big(\frac{1}{j+1}\binom{2j}{j}\Big) \\
&= \sum_{j=0}^{\infty}\frac{1}{j!(j+1)!}z^{2j},
\end{aligned}$$

as desired. $\blacksquare$

The strong version of Wigner's semi-circle law

Theorem. For each n in $\mathbb{N}$, let X_n be a random matrix in $\mathrm{GUE}(n, \frac{1}{n})$. Then there is a measurable set $S \subseteq \Omega$ with probability one, such that

$$\mu_{X_n(\omega)} \xrightarrow{\mathrm{w}} \tfrac{1}{2\pi}\sqrt{4-t^2}\,1_{[-2,2]}(t)\,\mathrm{d}t, \qquad \text{as } n \to \infty,$$

for all ω in S.
In other words, for any interval I in $\mathbb{R}$, we have

$$\frac{1}{n}\#\{j \in \{1,2,\dots,n\} \mid \lambda_j(X_n(\omega)) \in I\} \underset{n\to\infty}{\longrightarrow} \frac{1}{2\pi}\int_{I\cap[-2,2]} \sqrt{4-t^2}\,\mathrm{d}t,$$

for all ω in S.

Sketch of Proof.

Step 1 (Concentration inequality). Let $G_{N,\sigma}$ denote the Gaussian distribution on $\mathbb{R}^N$ with Lebesgue density

$$\frac{dG_{N,\sigma}(x)}{dx} = (2\pi\sigma^2)^{-N/2} \exp(-\tfrac{\|x\|^2}{2\sigma^2}),$$

where $\|x\|$ is the Euclidean norm of x. Furthermore, let $F\colon \mathbb{R}^N \to \mathbb{R}$ be a function that satisfies the Lipschitz condition

$$|F(x) - F(y)| \leq c\|x - y\|, \qquad (x, y \in \mathbb{R}^N), \tag{1}$$

for some positive constant c. Then for any positive number ϵ, we have that

$$G_{N,\sigma}\big(\{x \in \mathbb{R}^N \mid |F(x) - \mathbb{E}(F)| > \epsilon\}\big) \leq 2\exp\big(-\tfrac{K\epsilon^2}{c^2\sigma^2}\big),$$

where $\mathbb{E}(F) = \int_{\mathbb{R}^N} F(x)\,\mathrm{d}G_{N,\sigma}(x)$, and $K = \frac{2}{\pi^2}$.

Sketch of Proof (continued).

Step 2. Use (1) on the function

$$F(A) = \mathrm{tr}_n\big(f(A)\big), \qquad (A \in M_n(\mathbb{C})_{\mathrm{sa}}),$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition:

$$|f(y) - f(x)| \leq c|x - y|, \qquad (x, y \in \mathbb{R}).$$

This yields the estimate

$$P\Big(\big|\mathrm{tr}_n(f(X_n)) - \mathbb{E}\big\{\mathrm{tr}_n(f(X_n))\big\}\big| > \epsilon\Big) \leq \exp(-\tfrac{n^2 K \epsilon^2}{c^2}).$$

Step 3. Use the Borel-Cantelli lemma!

A differential equation for the spectral density h_n

Proposition. For any positive integer n, the spectral density h_n of a GUE random matrix X_n satisfies the differential equation:

$$\frac{1}{n^2} h_n'''(x) + (4 - x^2) h_n'(x) + x h_n(x) = 0.$$

Sketch of proof.

Step 1. We have seen that

$$\psi_n(z) := \int_{\mathbb{R}} \exp(zt) h_n(t)\,\mathrm{d}t = \mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} = \mathrm{e}^{z^2/2n}\eta_n(z^2/n), \qquad ($$

with $\eta_n\colon \mathbb{R} \to \mathbb{R}$ the function given by

$$\eta_n(s) = \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} s^j.$$

It is a classical result that η_n satisfies the differential equation:

$$s\eta_n''(s) + (2+s)\eta_n'(s) + (n-1)\eta_n(s) = 0. \qquad (2)$$

Sketch of proof (continued).

Step II. Using the substitution $s = z^2/n$ it follows from (2) that ψ satisfies the differential equation

$$n^2 z\psi_n''(z) + 3n^2\psi_n'(z) - (4n^2 z + z^3)\psi_n(z) = 0.$$

Setting $z = -\mathrm{i}y$ for y in $\mathbb{R}$, it follows that the Fourier transform $\widehat{h}_n(y) = \psi_n(-\mathrm{i}y)$ satisfies the differential equation:

$$n^2\mathrm{i}y\widehat{h}_n''(y) + 3n^2\mathrm{i}\widehat{h}_n'(y) + (4n^2\mathrm{i}y - \mathrm{i}y^3)\widehat{h}_n(y) = 0. \qquad (3)$$

Sketch of proof (continued).

Step III. Consider the co-Fourier transform $\overline{\mathcal{F}}\colon L_1(\mathbb{R}) \to L_1(\mathbb{R})$ defined by

$$[\overline{\mathcal{F}} f](y) = \int_{\mathbb{R}} \exp(\mathrm{i}yt) f(t)\,\mathrm{d}t, \qquad (y \in \mathbb{R}),$$

and recall that by Fourier inversion $\overline{\mathcal{F}}\widehat{h_n} = (2\pi)h_n$. Applying now $\overline{\mathcal{F}}$ to (3), we obtain that h_n satisfies

$$\frac{1}{n^2} h_n'''(x) + (4 - x^2) h_n'(x) + x h_n(x) = 0,$$

as desired. ■

The Harer-Zagier Recursion Formulae

Theorem. For each n in $\mathbb{N}$ and p in $\mathbb{N}_0$, put

$$\gamma(p, n) = \int_{\mathbb{R}} t^{2p} h_n(t)\,\mathrm{d}t = \mathbb{E}\{\mathrm{tr}_n(X_n^{2p})\},$$

where $X_n \in \mathrm{GUE}(n, \frac{1}{n})$. These moments satisfy the recursion formula

$$(p+2)\gamma(p+1, n) = \frac{(4p^2-1)p}{n^2}\gamma(p-1, n) + (4p+2)\gamma(p, n), \tag{4}$$

for any positive integer p.

Proof of the Harer-Zagier recursion formulae

Since h_n satisfies the differential equation

$$\frac{1}{n^2} h_n'''(x) + (4 - x^2) h_n'(x) + x h_n(x) = 0,$$

it follows by multiplication by x^{2p+1} and partial integration that

$$\begin{aligned} 0 &= \int_{\mathbb{R}} x^{2p+1} \left(n^{-2} h_n'''(x) + (4 - x^2) h_n'(x) + x h_n(x) \right) \mathrm{d}x \\ &= \int_{\mathbb{R}} \Big(- n^{-2} (2p+1)(2p)(2p-1) x^{2p-2} - 4(2p+1) x^{2p} \\ &\qquad\qquad + (2p+3) x^{2p+2} + x^{2p+2} \Big) h_n(x)\, \mathrm{d}x \\ &= \frac{-2(4p^2-1)p}{n^2} \gamma(p-1, n) - 4(2p+1)\gamma(p, n) + 2(p+2)\gamma(p+1, n), \end{aligned}$$

from which (4) follows readily. ■

Convergence of largest and smallest eigenvalue for a GUE matrix

Theorem. For each positive integer n, let X_n be a random matrix from $\mathrm{GUE}(n, \frac{1}{n})$, and let $\lambda_{\max}(X_n)$ and $\lambda_{\min}(X_n)$ denote, respectively, the largest and smallest eigenvalues of X_n. Then

$$\lim_{n\to\infty} \lambda_{\max}(X_n) = 2, \qquad \text{almost surely,}$$

and

$$\lim_{n\to\infty} \lambda_{\min}(X_n) = -2, \qquad \text{almost surely.}$$

Proof of $\limsup_{n\to\infty} \lambda_{\max}(X_n) \leq 2$ almost surely

It suffices to show that

$$\forall \epsilon > 0\colon\ P\Big(\limsup_{n\to\infty} \lambda_{\max}(X_n) \leq 2+\epsilon\Big) = 1,$$

which will follow if we show that

$$\forall \epsilon > 0\colon\ P\big(\lambda_{\max}(X_n) \leq 2+\epsilon,\ \text{for all sufficiently large } n\big) = 1,$$

which in turn follows from the Borel-Cantelli Lemma if we show that

$$\forall \epsilon > 0\colon\ \sum_{n=1}^{\infty} P\big(\lambda_{\max}(X_n) \geq 2+\epsilon\big) < \infty. \tag{5}$$

Proof of $\limsup_{n\to\infty} \lambda_{\max}(X_n) \leq 2$ (continued)

So let $\epsilon > 0$ be given, and then note that for any $t > 0$ we have by Chebychev's inequality that for any $t > 0$,

$$\begin{aligned} P\big(\lambda_{\max}(X_n) \geq 2+\epsilon\big) &= P\big(\exp(t\lambda_{\max}(X_n)) \geq \exp(t(2+\epsilon)\big) \\ &= \exp(-(2+\epsilon)t)\mathbb{E}\big\{\exp(t\lambda_{\max}(X_n))\big\}. \end{aligned} \tag{6}$$

Note here that

$$\begin{aligned} \exp(t\lambda_{\max}(X_n)) &= \lambda_{\max}(\exp(tX_n)) \\ &\leq \sum_{j=1}^{n} \lambda_j(\exp(tX_n)) = n\mathrm{tr}_n\big(\exp(tX_n)\big), \end{aligned}$$

since all the eigenvalues of $\exp(tX_n)$ are positive. Thus.....

$$\begin{aligned}
\mathbb{E}\Big\{\exp(t\lambda_{\max}(X_n))\Big\} &\le n\mathbb{E}\Big\{\mathrm{tr}_n\Big(\exp(tX_n)\Big)\Big\} \\
&= n\exp\left(\tfrac{t^2}{2n}\right)\sum_{j=0}^{n-1}\frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\left(\frac{t^2}{n}\right)^j. \\
&\le n\exp\left(\tfrac{t^2}{2n}\right)\sum_{j=0}^{\infty}\frac{n^j}{j!(j+1)!}\left(\frac{t^2}{n}\right)^j. \\
&\le n\exp\left(\tfrac{t^2}{2n}\right)\sum_{j=0}^{\infty}\left(\frac{t^j}{j!}\right)^2. \\
&\le n\exp\left(\tfrac{t^2}{2n}\right)\left(\sum_{j=0}^{\infty}\frac{t^j}{j!}\right)^2. \\
&= n\exp\left(\tfrac{t^2}{2n}+2t\right).
\end{aligned}$$

Proof of $\limsup_{n\to\infty} \lambda_{\max}(X_n) \le 2$ (continued)

Comparing with (6) we conclude that

$$\begin{aligned} P\big(\lambda_{\max}(X_n) \ge 2+\epsilon\big) &\le \exp(-(2+\epsilon)t)\mathbb{E}\big\{\exp(t\lambda_{\max}(X_n))\big\} \\ &\le n\exp(-(2+\epsilon)t)\exp\left(\tfrac{t^2}{2n}+2t\right) \\ &= n\exp\left(-\epsilon t+\tfrac{t^2}{2n}\right), \end{aligned}$$

which holds for all $t > 0$. Putting $t = n\epsilon$ we obtain

$$P\big(\lambda_{\max}(X_n) \ge 2+\epsilon\big) \le n\exp\left(\tfrac{-n\epsilon^2}{2}\right),$$

from which (5) follows immediately. ■

Proof of $\liminf_{n\to\infty} \lambda_{\max}(X_n) \geq 2$ almost surely

Let ϵ be a positive number. Then for almost all ω we have

$$\begin{aligned}\#\{j \in \{1,\dots,n\} \mid \lambda_j(X_n(\omega)) \geq 2-\epsilon\} &= \sum_{j=1}^{n} \delta_{\lambda_j(X_n(\omega))}([2-\epsilon,\infty)) \\ &= n\mu_{X_n(\omega)}([2-\epsilon,\infty)) \\ &\xrightarrow[n\to\infty]{} \infty,\end{aligned}$$

since, according to the strong version of Wigner's semi-circle law,

$$\mu_{X_n(\omega)}([2-\epsilon,\infty)) \xrightarrow[n\to\infty]{} \frac{1}{2\pi}\int_{2-\epsilon}^{2} \sqrt{4-t^2}\,\mathrm{d}t > 0,$$

for almost all ω. It follows in particular that

$$\liminf_{n\to\infty} \lambda_{\max}(X_n(\omega)) \geq 2-\epsilon, \qquad \text{for almost all } \omega.$$