10.3 - Estimating Sums

- In our study of sequences and series so far, we have discovered that it’s rather difficult to find the exact sum of a series. A first pass at trying to estimate a series is by brute force: sum up the first 1,000 or 100,000 terms to see “where the sum is headed.” Without the aid of a computer, however, this can become a nightmare.

What we can do, however, is to look at the associated improper integral. The Remainder Estimate for the Integral Test tells us that if \( \sum a_n \) converges by the Integral Test and \( R_n = s - s_n \) is the error from approximating the true sum \( s \) with \( s_n \), then

\[
\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx.
\]

- Given the series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \), let’s sum the first 10 terms to approximate the sum.

Then

\[
\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = 1 + \frac{1}{8} + \frac{1}{27} + \cdots + \frac{1}{1000} \approx 1.1975.
\]

Using the previous remainder estimate, we see that

\[
R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{R \to \infty} \left[ \int_{10}^{R} \frac{1}{x^3}\,dx \right] = \lim_{R \to \infty} \left[ -\frac{1}{2R^2} + \frac{1}{200} \right] = 0.005.
\]

Let’s say we want to have an estimate that’s ten times better: an estimate that gives an error of only 0.0005. How many terms would we have to sum to be sure to accomplish this?

We can show that \( \int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2} \) for any \( n \). Then we want

\[
R_n \leq \int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2} \Rightarrow \frac{1}{2n^2} < 0.0005.
\]

Solving this inequality, we get

\[
n^2 > 1000 \Rightarrow n > \sqrt{1000} \approx 31.6,
\]

so we would want to sum 32 terms to guarantee an error of less than 0.0005.

10.4 - Other Convergence Tests

- In this section, we wrap up our impressive array of convergence tests for series with a few additions. At the beginning of the section, we defined an alternating series to be a series whose terms are alternately positive and negative. The \( n \)th term of an alternating series is given by \( a_n = (-1)^{n-1}b_n \) or \( a_n = (-1)^nb_n \).

So, VERY informally, we can say that an alternating series is one whose partial sums \( s_n \) oscillate back and forth about its true sum \( s \).
Let’s consider the 50th partial sum $s_{50}$ of the alternating series given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}.$$ 

Will this partial sum be an overestimate or an underestimate of the total sum $s$?

Since $\frac{(-1)^{49}}{50^3}$ is negative (we’re taking an odd power of a negative number), $s_{50}$ is an underestimate of $s$. This can be easier seen by referring to Figure 1 on page 605 in the textbook.

Following from the notion of an alternating series was the appropriately-titled Alternating Series Test, which stated that an alternating series of the form $\sum (-1)^{n-1}b_n$ converges if the sequence $b_n$ decreases to 0.

Consider as an example the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}.$$ 

For what values of $p$ does this series converge? Let’s check the two conditions of the Alternating Series test: that $b_n$ is decreasing and that its limit is 0. For $p \leq 0$, we see that

$$\lim_{n \to \infty} \frac{(-1)^{n-1}}{n^p} = \text{DNE},$$

so the series diverges for $p \leq 0$ by the Divergence Test. For $p > 0$, we see that

$$\frac{1}{(n+1)^p} < \frac{1}{n^p},$$

so that shows that $b_n$ is decreasing. Also,

$$\lim_{n \to \infty} \frac{1}{n^p} = 0,$$

so, by the Alternating Series Test, this series converges for $p > 0$.

The last convergence test we learned about was the Ratio Test, which says that:

- If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, the series $\sum a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, the series $\sum a_n$ is divergent.
- If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive.

Let’s apply this to the following series:

$$\sum_{n=1}^{\infty} \frac{(n+2)!}{n!10^n}.$$
I think it’s plainly obvious that trying to integrate the related function to this, compare it to anything else, or find the limit of this would be a nightmare, so let’s try the Ratio Test. The Ratio Test turns out to be a good idea for many series that involve factorials and powers of \( n \). Let’s then consider

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+3)!}{(n+1)!(n+2)!} \cdot \frac{n^{10^n}}{(n+1)^n} = \frac{1}{10} \lim_{n \to \infty} \frac{n+3}{n+1} = \frac{1}{10} < 1,
\]

so the series is absolutely convergent (and thus convergent) by the Ratio Test.

10.5 - Power Series

- A power series is a series of the form

\[
\sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,
\]

where \( x \) is a variable and the \( c_n \)'s are constants called the coefficients of the series. We get a power series centered at \( x = a \) if we replace \( x \) by \( (x-a) \).

We discovered in this section that, for a given power series \( \sum_{n=0}^{\infty} c_n (x-a)^n \), there are only three possibilities:

- The series converges only when \( x = a \).
- The series converges for all \( x \).
- There is a positive number \( R \) (known as the radius of convergence) such that the series converges if \( |x-a| < R \) and diverges if \( |x-a| > R \).

- Suppose, then, that we know \( \sum_{n=6}^{\infty} c_n x^n \) converges when \( x = -4 \) and diverges when \( x = 6 \). We then know that the radius of convergence \( R \) of this series is in the range \( 4 < R < 6 \), although we can’t know exactly what it is. This implies that the series certainly converges for \( x \in (-4, 4) \), that it might converge for \( x \in (-6, 6) \), and that it certainly doesn’t converge for \( |x| \geq 6 \). Given this information, we can discover a few things about related series:

  - The series \( \sum_{n=1}^{\infty} c_n \) [converges] since \( x = 1 \) here and 1 lies in the interval \((-4, 4)\).
  - The series \( \sum_{n=1}^{\infty} c_n 8^n \) [diverges] since 8 is not in the interval \((-6, 6)\).
  - The series \( \sum_{n=1}^{\infty} c_n (-3)^n \) [converges] since \(-3\) is in the interval \((-4, 4)\).
  - The series \( \sum_{n=1}^{\infty} (-1)^n c_n 9^n \) [diverges] since neither \( 9 \) nor \(-9\) is in the interval \((-4, 4)\).
To determine the radius of convergence $R$ of a power series, we apply the Ratio Test and see which values of $x$ guarantee convergence. For example, consider the series
\[ \sum_{n=1}^{\infty} \frac{(x-4)^n}{n5^n}. \]
Applying the Ratio Test to this series:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \frac{|x-4|^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{|(x-4)^n|} \right) = \frac{1}{5}|x-4| \lim_{n \to \infty} \frac{n}{n+1} = \frac{1}{5}|x-4|.
\]
By the Ratio Test, we want this quantity to be less than 1 for convergence, so
\[
\frac{1}{5}|x-4| < 1 \Rightarrow |x-4| < 5 \Rightarrow -5 < x - 4 < 5 \Rightarrow -1 < x < 9.
\]
From this, we see that the radius of convergence is $R = 5$ and the interval of convergence is $-1 < x < 9$. 