EQUIDISTRIBUTION OF GROSS POINTS OVER RATIONAL FUNCTION FIELDS

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Abstract. In this paper, we prove a quantitative equidistribution theorem for Gross points over the rational function field \( \mathbb{F}_q(t) \). We apply this result to prove that the reduction map from CM Drinfeld modules to supersingular Drinfeld modules is surjective. Our proofs rely crucially on a period formula of F.-T. Wei and J. Yu, and a Lindel"of-type bound for central values of Rankin-Selberg \( L \)-functions associated to twists of automorphic forms of Drinfeld-type by ideal class group characters.

1. Introduction and statement of results

In this paper, we prove a quantitative equidistribution theorem for Gross points over the rational function field \( \mathbb{F}_q(t) \) (see Theorem 1.3 and Corollary 1.4). We then use this equidistribution theorem to prove that the reduction map from CM Drinfeld modules to supersingular Drinfeld modules is surjective (see Theorem 1.6).

1.1. Gross points over \( \mathbb{F}_q(t) \). In this section, we develop some background we will need concerning Gross points over \( \mathbb{F}_q(t) \), following closely the discussion in Gross [Gr], Wei and Yu [WY, §§ 1.1–2].

Let \( q \in \mathbb{Z} \) be a power of an odd prime, \( A = \mathbb{F}_q[t] \) be the polynomial ring in a variable \( t \), and \( k = \mathbb{F}_q(t) \) be the rational function field. We let \( k_\infty = \mathbb{F}_q((t^{-1})) \) be the completion of \( k \) at its infinite place. Let \( P_0 \in A \) be a monic irreducible polynomial of degree \( \deg(P_0) \geq 3 \). Let \( D \in A \) be irreducible of odd degree, in particular ensuring that \( K = k(\sqrt{D}) \) is imaginary quadratic, i.e., \( \infty \) does not split in \( K \). Assume further that \( P_0 \) is inert in \( K \), and so \( \chi_K(P_0) = -1 \), where \( \chi_K \) is the quadratic character associated to \( K \). Let \( \mathcal{O}_D = A[\sqrt{D}] \) be the ring of integers, \( \text{Pic}(\mathcal{O}_D) \) be the class group, and \( h(D) \) be the class number of \( K \), respectively.

Let \( B \) be the quaternion algebra over \( k \) which is ramified at \( P_0 \) and \( \infty \), and fix a maximal \( A \)-order \( R \) of \( B \). There are finitely many equivalences classes of left ideals of \( R \), and the number \( n \) of such classes is called the class number of \( R \). Let \( \{I_i\}_{i=1}^n \) be a set of representatives of these equivalence classes. One associates to each \( I_i \) the maximal right \( A \)-order of \( B \) defined by

\[
R_i := \{x \in B : I_i x \subseteq I_i\}.
\]

Given a finite place \( v \) of \( k \), let \( k_v \) be the completion of \( k \) at \( v \), let \( A_v \) be the closure of \( A \) in \( k_v \), let \( \hat{k} \) be the finite adele ring of \( k \), and let \( \hat{A} = \prod_v A_v \). Also, let \( \hat{R} = R \otimes_A \hat{A} \) and \( \hat{B} = B \otimes_k \hat{k} \).

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The left ideal classes of $R$ correspond to the orbits of $B^\times$ acting on the right of $\hat{R}^\times \backslash \hat{B}^\times$, so that the class number $n$ is the number of double cosets,

$$n = [\hat{R}^\times \backslash \hat{B}^\times / B^\times].$$

The choice of representative ideals $\{I_i\}_{i=1}^n$ corresponds to a choice of coset representatives $\{g_i\}_{i=1}^n$ in $\hat{R}^\times \backslash \hat{B}^\times$ such that

$$\hat{B}^\times = \bigcup_{i=1}^n \hat{R}^\times g_i B^\times. \quad (1)$$

The right order $R_i$ is given by

$$R_i = B \cap g_i^{-1} \hat{R} g_i.$$

An optimal embedding $f : O_D \hookrightarrow R_i$ of $O_D$ into $R_i$ is a field embedding $f : K \to B$ which satisfies

$$f(K) \cap g_i^{-1} \hat{R} g_i = f(O_D)$$
in $\hat{B}$. The group $B^\times$ acts on the right of the set

$$\hat{R}^\times \backslash \hat{B}^\times \times \text{Hom}(K, B)$$
by

$$(g, f) \mapsto (gb, b^{-1} fb).$$

There is a bijection between the set of all optimal embeddings of $O_D$ into the $n$ orders $R_i$, modulo conjugation by $R_i^\times$, and the classes $(g, f) \mod B^\times$ in the quotient space

$$(\hat{R}^\times \backslash \hat{B}^\times \times \text{Hom}(K, B))/B^\times \quad (2)$$
satisfying

$$f(K) \cap g^{-1} \hat{R} g = f(O_D). \quad (3)$$

The quotient space (2) can be interpreted geometrically as the set of $K$-points of a definite Shimura curve $X_{P_0}$ defined over $k$ as follows. There is a genus zero curve $Y$ defined over $k$ associated to $B$ whose points over $K$ are

$$Y(K) = \{y \in B \otimes_k K : y \neq 0, \text{ Tr}(y) = N(y) = 0\}/K^\times.$$

The curve $Y$ can be described explicitly as a conic in $\mathbb{P}^2$. The group $B^\times$ acts on $Y$ on the right by conjugation. Moreover, $Y(K)$ can be canonically identified with the set of embeddings $\text{Hom}(K, B)$. Define the definite Shimura curve $X_{P_0}$ by the double coset space

$$X_{P_0} := (\hat{R}^\times \backslash \hat{B}^\times \times Y)/B^\times.$$

The decomposition (1) gives a bijection to a disjoint union of genus 0 curves,

$$X_{P_0} \to \bigsqcup_{i=1}^n X_i$$
defined by

$$(\hat{R}^\times g_i, y) \mod B^\times \mapsto y \mod R_i^\times,$$
where $X_i := Y/R_i^\times$ and $R_i^\times = g_i^{-1} \hat{R}^\times g_i^{-1} \cap B^\times$ is a finite group for $i = 1, \ldots, n$. 
Definition 1.1. A Gross point of discriminant $D$ on $X_{P_0}(K)$ is a point $x = (g, y)$ in the image

$$\text{Image} \left[ \hat{R}^\times \backslash \hat{B}^\times \times Y(K) \to X_{P_0}(K) \right]$$

such that the embedding $f \in \text{Hom}(K, B)$ corresponding to the component $y \in Y(K)$ of $x$ satisfies (3). If the component $g \in \hat{R}^\times \backslash \hat{B}^\times / B^\times$ of $x$ is congruent to the double coset representative $g_i$, then $x$ lies on the component $X_i(K)$ of $X_{P_0}(K)$.

Let $\text{Gr}_{D, P_0}$ denote the set of Gross points of discriminant $D$. There is an action of the group $\text{Pic}(\mathcal{O}_D)$ on $\text{Gr}_{D, P_0}$ given as follows. Let $\hat{K} = K \otimes_k \hat{k}$ and $\hat{\mathcal{O}}_D = \mathcal{O}_D \otimes_A \hat{A}$. Then

$$\text{Pic}(\mathcal{O}_D) \cong \hat{\mathcal{O}}_D^\times \backslash \hat{K}^\times / K^\times.$$  

Let $x = (g, y)$ be a Gross point of discriminant $D$, and let $f \in \text{Hom}(K, B)$ be the embedding corresponding to the component $y$. This embedding induces a homomorphism $\hat{f} : \hat{K}^\times \to \hat{B}^\times$. Let $a \in K^\times$ and define

$$x_a := (g \hat{f}(a), y).$$

This gives a free action of $\text{Pic}(\mathcal{O}_D)$ on $\text{Gr}_{D, P_0}$ which divides the set of Gross points into two simple transitive orbits of size $h(D)$; in particular, $\# \text{Gr}_{D, P_0} = 2h(D)$ (see e.g. [Gr, p. 133] and [WY, Lem. 1.4]). We denote this action by $x \mapsto x^\sigma$ for $\sigma \in \text{Pic}(\mathcal{O}_D)$.

The action of $\text{Pic}(\mathcal{O}_D)$ on the set of Gross points of discriminant $D$ also translates to an action of $\text{Pic}(\mathcal{O}_D)$ on the corresponding optimal embeddings. To describe this, we follow the first paragraph of [Gr, p. 134].

Let $a$ be the ideal (projective module of rank 1 in $K$) which is determined by the ideal $a \mod \hat{\mathcal{O}}^\times_D$; specifically, $a = K \cap a\mathcal{O}_D$. Let $R'_{i,a}$ be the right order of the left $R_i$-module $R_i a$. More precisely, since $R_i$ is a maximal right order of $B$, we can consider the set of $n$ equivalence classes of left $R_i$-ideals in $B$. Then, viewing $R_i a$ as a left $R_i$-ideal, we can (as above) associate to $R_i a$ a maximal right order $R'_{i,a}$ of $B$ defined by

$$R'_{i,a} = \{ b \in B : R_i ab \subseteq R_i a \}. \quad (4)$$

Since $\mathcal{O}_D$ also acts on the right of $a$, the optimal embedding $f : \mathcal{O}_D \to R_i$ corresponding to the Gross point $x$ induces an optimal embedding $f' : \mathcal{O}_D \to R'_{i,a}$ corresponding to the Gross point $x_a$. We denote the embedding $f'$ by $f^\sigma$ and the maximal order $R'_{i,a}$ by $R'_{i,\sigma}$, where $\sigma \in \text{Pic}(\mathcal{O}_D)$ corresponds to the class of $a$ in $\text{Pic}(\mathcal{O}_D)$.

1.2. Equidistribution of Gross points over $\mathbb{F}_q(t)$. As we have seen, the Shimura curve $X_{P_0}$ is the disjoint union of $n$ genus zero curves $X_i$ defined over $k$. Let $\text{Pic}(X_{P_0})$ denote the Picard group of $X_{P_0}$. If $e_i$ denotes the class of degree 1 in $\text{Pic}(X_{P_0})$ corresponding to the component $X_i$, then we have

$$\text{Pic}(X_{P_0}) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n.$$  

Since a Gross point $x \in \text{Gr}_{D, P_0}$ lies on a component $X_i$, it determines a class $e_x$ in $\text{Pic}(X_{P_0})$ which for notational convenience we continue to denote by $x$.

For each $i$, we let $w_i := \#(R_i^\times)/(q-1)$, and we define a probability measure on the set of divisor classes $\mathcal{S} := \{e_i\}_{i=1}^n$ by

$$\mu_{P_0}(e_i) := \frac{w_i^{-1}}{\sum_{j=1}^n w_j^{-1}}.$$
We note that Gekeler [Ge1, Satz (5.9)] has proved the mass formula

$$\sum_{j=1}^{n} \frac{1}{w_j} = \frac{q^{\deg(P_0)} - 1}{q^2 - 1},$$

which shows how the total mass grows with the degree of $P_0$. Furthermore, for each $i$ we have $w_i$ is either 1 or $q + 1$.

Now given a Gross point $x = x_{0,D} \in \text{Gr}_{D,P_0}$ and a subgroup $G_D < \text{Pic}(\mathcal{O}_D)$, define the $G_D$-orbit

$$G_D \cdot x_{0,D} := \{ x_{0,D}^\sigma : \sigma \in G_D \} \subseteq S.$$

We want to study the distribution of the sequence of orbits $G_D \cdot x_{0,D}$ on $S$ with respect to the probability measure $\mu_{P_0}$ as $\deg(D) \to \infty$. To make sense of this distribution problem, we must have

$$\#(G_D \cdot x_{0,D}) \to \infty$$

as $\deg(D) \to \infty$. In particular, for fixed $P_0$, the size of the orbit $G_D \cdot x_{0,D}$ will eventually exceed the number $n = n(P_0)$ of divisor classes $e_i$, which (as we have seen) equals the class number of the maximal order $R$ in $B$. As shown in the following result, the condition (6) is ensured by a suitable bound on the index of $G_D$ in $\text{Pic}(\mathcal{O}_D)$.

**Proposition 1.2.** Fix an absolute constant $0 \leq \eta < 1/2$ and let $G_D < \text{Pic}(\mathcal{O}_D)$ be a subgroup of index satisfying

$$[\text{Pic}(\mathcal{O}_D) : G_D] \leq \|D\|^\eta$$

where $\|D\| := q^{\deg(D)}$. Then

$$\#(G_D \cdot x_{0,D}) \gg \varepsilon q^{-1} \|D\|^{(\frac{1}{2} - \eta) - \varepsilon}$$

where the implied constant is effective. In particular, $\#(G_D \cdot x_{0,D}) \to \infty$ as $\deg(D) \to \infty$.

**Proof.** Let $L(\chi_K, s)$ be the $L$-function of the quadratic character $\chi_K$. Then we have the class number formula (see e.g. [CWY] §2.2)

$$L(\chi_K, 1) = q\|D\|^{-1/2}h(D).$$

By work of Weil, the Riemann hypothesis is known for the $L$-function $L(\chi_K, s)$, which allows one to prove the following effective Siegel-type bound (see e.g. [AT] Lem. 3.3)

$$L(\chi_K, 1) \gg \varepsilon \|D\|^{-\varepsilon}$$

for any $\varepsilon > 0$. This yields the following effective lower bound for the class number,

$$h(D) \gg \varepsilon q^{-1} \|D\|^{\frac{1}{2} - \varepsilon}.$$  (8)

In particular, from (7) and (8) we get

$$\#(G_D \cdot x_{0,D}) = |G_D| \frac{h(D)}{[\text{Pic}(\mathcal{O}_D) : G_D]} \gg \frac{h(D)}{\|D\|^\eta} \gg \varepsilon q^{-1} \|D\|^{(\frac{1}{2} - \eta) - \varepsilon}.$$

We will prove that if $0 \leq \eta < 1/4$ and

$$[\text{Pic}(\mathcal{O}_D) : G_D] \leq \|D\|^\eta,$$

then the sequence of orbits $G_D \cdot x_{0,D}$ becomes quantitatively equidistributed on $S$ with respect to $\mu_{P_0}$ as $\deg(D) \to \infty$. We will deduce our equidistribution results as a consequence of the following theorem.
Theorem 1.3. Let \(q \in \mathbb{Z}\) be a power of an odd prime, \(A = \mathbb{F}_q[t]\) be the polynomial ring in a variable \(t\), and \(k = \mathbb{F}_q(t)\) be its fraction field. Let \(P_0 \in A\) be a monic irreducible polynomial satisfying \(\deg(P_0) \geq 3\). Let \(D \in A\) be an irreducible polynomial of odd degree such that \(P_0\) is inert in the imaginary quadric field \(K = k(\sqrt{D})\). Given a subgroup \(G_D < \text{Pic}(\mathcal{O}_D)\), a Gross point \(x = x_{0,D} \in \text{Gr}_{D,P_0}\), and a divisor class \(e_i \in S\), define the counting function

\[
N_{G_D,P_0,e_i} := \#\{\sigma \in G_D : x_{0,D}^\sigma = e_i\}.
\]

Then for all \(\varepsilon > 0\), we have

\[
\frac{N_{G_D,P_0,e_i}}{|G_D|} = \mu_{P_0}(e_i) + O_\varepsilon \left( |\text{Pic}(\mathcal{O}_D) : G_D| q^{5/4} \|P_0\|^{1/2+\varepsilon} \|D\|^{-1/4+\varepsilon} \right)
\]

where the implied constant in the error term is uniform in \(e_i\) and effective.

The following corollary is now an immediate consequence of Theorem 1.3.

Corollary 1.4. Let notation and assumptions be as in Theorem 1.3. Fix an absolute constant \(0 \leq \eta < 1/4\) and let \(G_D < \text{Pic}(\mathcal{O}_D)\) be a subgroup of index satisfying

\[
|\text{Pic}(\mathcal{O}_D) : G_D| \leq \|D\|^\eta.
\]

Then the sequence of orbits \(G_D \cdot x_{0,D}\) becomes quantitatively equidistributed on \(S\) with respect to \(\mu_{P_0}\) as \(\deg(D) \to \infty\). In particular, we have

\[
\frac{N_{G_D,P_0,e_i}}{|G_D|} = \mu_{P_0}(e_i) + O_\varepsilon \left( q^{5/4} \|P_0\|^{1/2+\varepsilon} \|D\|^{-(1/4-\eta)+\varepsilon} \right). \tag{9}
\]

Remark 1.5. Michel and Venkatesh [MV] developed a general framework concerning sparse equidistribution problems for toric orbits of special points on Shimura varieties constructed from quaternion algebras over totally real fields. These problems have been solved in a wide range of settings. For example, Michel [M] proved sparse equidistribution of Gross points corresponding to supersingular elliptic curves. Theorem 1.3 can be viewed as part of this framework, in which the quaternion algebra is definite and the base field is the rational function field \(k = \mathbb{F}_q(t)\).

1.3. Supersingular reduction of CM Drinfeld modules. In this section, we define the reduction map from CM Drinfeld modules to supersingular Drinfeld modules, and state our result showing that this map is surjective if \(\deg(D)\) is sufficiently large (in a quantitative sense which will be made precise). We first recall some definitions about Drinfeld modules (see [Go] Ch. 4, [T] Ch. 3 for more details).

Given a field \(L/\mathbb{F}_q\), the \(q\)-th power Frobenius endomorphism \(\tau : L \to L\) generates an \(\mathbb{F}_q\)-subalgebra of the endomorphism algebra of the additive group of \(L\), which we denote by \(L\{\tau\}\). The ring \(L\{\tau\}\) is the ring of twisted polynomials in \(\tau\) with coefficients in \(L\), and it is subject to the relation \(\tau c = \epsilon^q \tau\) for all \(c \in L\). Fixing an \(\mathbb{F}_q\)-algebra homomorphism \(\iota : A \to L\), a Drinfeld module of rank \(r\) over \(L\) is an \(\mathbb{F}_q\)-algebra homomorphism

\[
\phi : A \to L\{\tau\},
\]

so that for all \(a \in A\),

\[
\phi_a = \iota(a) + b_1 \tau + \cdots + b_{\deg(a)} \tau^{r \deg(a)}, \quad b_{\deg(a)} \neq 0.
\]
For two Drinfeld modules $\phi$, $\psi$ over $L$, a morphism $u : \phi \rightarrow \psi$ over $L$ is a twisted polynomial $u \in L\{\tau\}$, so that

$$u\phi_a = \psi_a u, \quad \forall a \in A.$$ 

If $\phi$ and $\psi$ have different ranks, then the only possible morphism between them is the zero morphism.

Henceforth, we will primarily focus on Drinfeld modules of rank 2. Since a Drinfeld module $\phi$ is completely determined by its value $\phi_t$, we can fix a rank 2 Drinfeld module over $L$ by setting,

$$\phi_t := \iota(t) + g\tau + \Delta\tau^2, \quad g, \Delta \in L,$$

in which case the $j$-invariant $j(\phi) := g^{q+1}/\Delta$ is an isomorphism invariant of $\phi$ over an algebraic closure $\overline{L}$.

Let $\mathcal{D}(O_D)$ be the set of $\overline{k}$-isomorphism classes of Drinfeld modules of rank 2 over $\overline{k}$ with complex multiplication (CM) by $O_D$. Here we take $\iota : A \hookrightarrow \overline{k}$ to be the inclusion map, and a Drinfeld module $\phi$ of rank 2 has complex multiplication by $O_D$ if there is an extension of $[\phi : O_D \rightarrow L\{\tau\}]$, coinciding with $\phi$ on $A$. By the theory of complex multiplication [Go Ch. 7], each isomorphism class in $\mathcal{D}(O_D)$ is represented by a sign-normalized Drinfeld-Hayes module (of rank 1 as a Drinfeld module, see [Go Ch. 7], [H]). Such a Drinfeld-Hayes module is defined over the Hilbert class field $H$ of $K$, i.e. the maximal abelian extension of $K$ that is everywhere unramified and totally split at the unique infinite place of $K$. The group $\text{Gal}(H/K) \cong \text{Pic}(O_D)$ acts simply transitively on $\mathcal{D}(O_D)$. In particular, given a Drinfeld-Hayes module $\phi \in \mathcal{D}(O_D)$, we have

$$\mathcal{D}(O_D) = \{[\phi^\sigma] : \sigma \in \text{Pic}(O_D)\},$$

where $\phi^\sigma$ denotes the action of $\text{Pic}(O_D)$ on $\phi$. There are $h(D)$ such isomorphism classes [H, Cor. 5.13].

Moreover, $\phi$ is defined over the ring of integers $O_H$ of $H$, so that

$$\phi_t = \iota(t) + b_1\tau + \tau^2, \quad b_1 \in O_H.$$ 

Thus if $\mathfrak{P}$ is a prime ideal of $O_H$ and $\overline{\iota} : A \rightarrow O_H/\mathfrak{P}$ is the map induced by the inclusion $A \hookrightarrow O_H$, we can define the reduction $\overline{\phi}$ of $\phi$,

$$\overline{\phi}_t = \overline{\iota}(t) + \overline{b}_1\tau + \tau^2, \quad \overline{b}_1 \in O_H/\mathfrak{P}, \quad \overline{b}_1 \equiv b_1 \pmod{\mathfrak{P}},$$

which is a Drinfeld module of rank 2 over $O_H/\mathfrak{P}$. Letting $P_0 \in A$ be the unique monic generator of $\mathfrak{P} \cap A$, we say that the reduction $\overline{\phi}$ is supersingular if

$$\overline{\phi}_{P_0} = \tau^{2\deg(P_0)} \in (O_H/\mathfrak{P})\{\tau\},$$

which is equivalent to $\overline{\phi}_{P_0}$ being purely inseparable as a map on $\mathbb{G}_a$ (see [Ge2 §4]). An equivalent condition is simply to check that the coefficient of $\tau^{\deg(P_0)}$ in $\phi_{P_0}$ is divisible by $P_0$, and this can be determined effectively (e.g., see [EP Cor 8.2]). Moreover, one can determine supersingular $j$-invariants via recursive identities on Drinfeld modular forms, using a construction of Cornelissen [Co].

If $\phi$ has supersingular reduction at $P_0 \nmid D$, then the endomorphism algebra of $\overline{\phi}$ is a maximal order in the quaternion algebra $B$ [Ge2 Thm. 2.9, Thm. 4.3]. Moreover, as this induces an embedding $O_D \hookrightarrow B$, it follows that $P_0$ must be inert in $O_D$ [Pa, Lem. 2.2].
Fix now $P_0 \in A$ monic, irreducible, and inert in $K$. We set $\mathbb{F}_{P_0} := A/(P_0)$ and let $D^{ss}(\mathbb{F}_{P_0})$ be the set of isomorphism classes of supersingular Drinfeld modules over $\mathbb{F}_{P_0}$. There are $n$ such isomorphism classes

$$D^{ss}(\mathbb{F}_{P_0}) = \{e_1, \ldots, e_n\},$$

and we further have bijective correspondences

$$\{X_1, \ldots, X_n\} \leftrightarrow \{e_1, \ldots, e_n\} \leftrightarrow \{I_1, \ldots, I_n\} \leftrightarrow \{R_1, \ldots, R_n\}$$

such that $\text{End}(e_i) \cong R_i$ (see [Ge2, §§3–4], [Pa, Thm. 2.6]). Letting $\mathfrak{p}$ be a prime above $P_0$ in $H$, we obtain a reduction map

$$r_{\mathfrak{p}} : D(O_D) \rightarrow D^{ss}(\mathbb{F}_{P_0}).$$

**Theorem 1.6.** The reduction map $r_{\mathfrak{p}}$ is surjective if $\|D\| \gg_{\varepsilon, q} \|P_0\|^{6+\varepsilon}$, where the implied constant in $\gg_{\varepsilon, q}$ is effective.

**Remark 1.7.** Liu, Young, and the second author [LMY, Cor. 1.3] proved a similar result for the reduction map from CM elliptic curves to supersingular elliptic curves.

### 2. Deducing Theorem 1.6 from Theorem 1.3

In this section, we show how Theorem 1.6 follows from Theorem 1.3 and Corollary 1.4. To do this we show that the image $r_{\mathfrak{p}}(D(O_D))$ can be identified with the set of Gross points of discriminant $D$. Then since Corollary 1.4 implies that this image becomes equidistributed among the classes $D^{ss}(\mathbb{F}_{P_0})$ as $\text{deg}(D) \rightarrow \infty$, every class $e \in D^{ss}(\mathbb{F}_{P_0})$ is hit at least once if $\text{deg}(D)$ is sufficiently large, in a quantitative sense which can be made precise using (9).

Since each class in $D(O_D)$ is represented by a sign-normalized Drinfeld-Hayes module $\phi$, if $r_{\mathfrak{p}}(\phi) = e$, then one obtains an embedding of the corresponding endomorphism rings,

$$f_{\mathfrak{p}, \phi} : \text{End}_H(\phi) \cong O_D \hookrightarrow \text{End}_{\mathbb{F}_{P_0}}(e) \cong R_e. \quad (10)$$

The induced embedding $f_{\mathfrak{p}, \phi} : K \hookrightarrow B$ is necessarily optimal. Indeed, this amounts to having the equality of subalgebras of $B$,

$$f_{\mathfrak{p}, \phi}(K) \cap g_e^{-1}\hat{R}g_e = f_{\mathfrak{p}, \phi}(O_D),$$

where $g_e$ is the coset representative in $\hat{R} \setminus \hat{B}$ corresponding to the ideal $I_e$; the left-hand side is always contained in the right, and by construction via sign-normalized modules, the right-hand side is contained in the left. We note that the optimality of embeddings in the context of CM elliptic curves was proved by Bertolini and Darmon [BD, Prop. 4.1].

Now recall that the $R_e \setminus \hat{B}$-conjugacy class of the optimal embedding $f_{\mathfrak{p}, \phi}$ corresponds to a Gross point $x_{\mathfrak{p}, \phi} = (g_e, f_{\mathfrak{p}, \phi})$ of discriminant $D$, and using the Pic($O_D$)-action on Gross points described §1.1 the $(R_e, \sigma) \setminus \hat{B}$-conjugacy class of the optimal embedding $f_{\mathfrak{p}, \sigma}$ for $\sigma \in \text{Pic}(O_D)$ corresponds to the Gross point $x_{\mathfrak{p}, \sigma} = (g_e, f_{\mathfrak{p}, \sigma})$ of discriminant $D$. We are now led to the following crucial equivariance result.

**Proposition 2.1.** Given a sign-normalized Drinfeld-Hayes module $\phi \in D(O_D)$, we have $f_{\mathfrak{p}, \phi} = f_{\mathfrak{p}, \phi^\sigma}$ for $\sigma \in \text{Pic}(O_D)$. 

Proof. Bertolini and Darmon [BD, Lem. 4.2] proved this result in the context of elliptic curves. We first recall the definition of $\phi^\sigma$: by identifying $\sigma$ with an element of $\text{Gal}(H/K)$, if $\phi_a = a + b_1 \tau + \cdots + b_\ell \tau^\ell$, then

$$
\phi_a^\sigma := a + b_1^\sigma \tau + \cdots + b_\ell^\sigma \tau^\ell.
$$

It is clear that $\phi^\sigma$ is also a sign-normalized Drinfeld-Hayes module defined over $O_H$.

The definition of $\phi^\sigma$ is compatible with an action of $\text{Pic}(O_D)$ in the following way. As originally defined by Hayes [Go, §4.9], [H, §§3–5], for a fixed integral ideal $a \subseteq O_D$ we let

$$
\mu_a := \text{unique monic generator of the left ideal } H\{\tau\} \cdot \phi(a) \text{ of } H\{\tau\},
$$

which is well-defined since $H\{\tau\}$ is a left principal ideal domain. Then there is a unique sign-normalized Drinfeld module $\phi^a$ such that $\mu_a : \phi \to \phi^a$ is a morphism, and furthermore $\mu_a \in O_H\{\tau\}$ by [H, Prop. 7.5]. If the class of $a$ in $\text{Pic}(O_D)$ corresponds to the Galois element $\sigma \in \text{Gal}(H/K)$ via class field theory, then by [H, Prop. 8.1],

$$
\phi_a^\sigma = \phi_a^\nu, \quad \forall a \in A.
$$

Now suppose that $r_\Psi(\phi) = e$. For simplicity, we will write $F = F_{P_0}$, and we will assume that $R_e = \text{End}_F(e) \subseteq F\{\tau\}$. Thus the embedding $f_{\Psi, \phi} : O_D \hookrightarrow R_e$ defined in [10] takes values in $F\{\tau\}$. If we fix again an integral ideal $a \subseteq O_D$, we take

$$
I = R_e \cdot f_{\Psi, \phi}(a) = R_e \cdot e(a),
$$

which is a left ideal of $R_e$. We let

$$
\nu_a := \text{unique monic generator of the left ideal } F\{\tau\} \cdot I \text{ of } F\{\tau\},
$$

and by J.-K. Yu [Y, §2], there is a unique Drinfeld module $e^I$ over $F$ such that $\nu_a : e \to e^I$ is a morphism. Moreover, by [Y, Prop. 2] we have

$$
\text{End}_F(e^I) = \{b \in B : Ib \subseteq I\} = R'_{e,a} = R'_{e,\sigma},
$$

the right order of $I$ in $B$, where $R'_{e,a}$ is defined in [4].

We claim that for our fixed ideal $a \subseteq O_D$, we have

$$
\nu_a \equiv \mu_a \pmod{\Psi},
$$

i.e., $\nu_a$ is obtained from $\mu_a$ by reducing each of its coefficients modulo $\Psi$. Letting $\overline{\mu}_a$ denote reduction of $\mu_a$ modulo $\Psi$, we observe that $\overline{\mu}_a \in e(a) \subseteq I$, since $r_\Psi(\phi) = e$. Now the zeros of $\mu_a$, as a function on $\overline{B}$, comprise the $a$-torsion of $\phi$ (see [Go, §4.9]), and as such

$$
\text{deg}_\tau(\mu_a) = \text{dim}_{F,a}(O_D/a).
$$

By the same token, every element of $I$ must vanish at the $a$-torsion of $e$, and so

$$
\text{deg}_\tau(\nu_a) \geq \text{dim}_{F,a}(O_D/a).
$$

Since $\overline{\mu}_a \neq 0$, it follows that $\overline{\nu}_a = \nu_a$. As a consequence, we see that

$$
r_\Psi(\phi^a) = r_\Psi(\phi^\sigma) = e^I.
$$

Therefore,

$$
f_{\Psi, \phi^a} : O_D \hookrightarrow \text{End}_F(e^I) = R'_{e,\sigma},
$$

but this coincides with $f_{\Psi, \phi}$ defined in [1.1] \hfill $\square$.
Proof of Theorem 1.6. By the preceding discussions, we have bijective correspondences
\[ r_{\mathfrak{F}}(D(\mathcal{O}_D)) \leftrightarrow \{ r_{\mathfrak{F}}(\phi^\sigma) : \sigma \in \text{Pic}(\mathcal{O}_D) \} \]
\[ \leftrightarrow \{ f_{\mathfrak{F},\phi^\sigma} : \sigma \in \text{Pic}(\mathcal{O}_D) \} \]
\[ \leftrightarrow \{ f_{\mathfrak{F},\phi}^\sigma : \sigma \in \text{Pic}(\mathcal{O}_D) \} \]
\[ \leftrightarrow \{ x_{\mathfrak{F},\phi}^\sigma : \sigma \in \text{Pic}(\mathcal{O}_D) \}. \]

Since the components \( \{X_1, \ldots, X_n\} \) of the definite Shimura curve \( X_{P_0} \), and hence the divisor classes \( S \), are identified with the isomorphism classes of supersingular Drinfeld modules \( \{e_1, \ldots, e_n\} \), and by Corollary 1.4 the Pic(\( \mathcal{O}_D \))-orbit \( \{x_{\mathfrak{F},\phi}^\sigma : \sigma \in \text{Pic}(\mathcal{O}_D)\} \) of the Gross point \( x_{\mathfrak{F},\phi} \) becomes equidistributed with respect to the probability measure \( \mu_{P_0} \) on \( S \) as \( \deg(D) \to \infty \), it follows from the above correspondences that the reduction map \( r_{\mathfrak{F}} \) is surjective if \( \deg(D) \) is sufficiently large. More precisely, since
\[ \|D\|^{1/2-\varepsilon} \ll_{\varepsilon, \eta} \mu_{\eta}(D) \ll_{\varepsilon, \eta} \|D\|^{1/2+\varepsilon} \]
and \( \mu_{P_0}(e_i) \propto \|P_0\|^{-1} \) (see [3]), then by (9) with \( \eta = 0 \) we have \( N_{\text{Pic}(\mathcal{O}_D),P_0,\varepsilon} > 0 \) (and hence that \( r_{\mathfrak{F}} \) is surjective) if \( \|D\| \gg_{\varepsilon, \eta} \|P_0\|^{6+\varepsilon} \), and we are done. \( \square \)

3. Rankin-Selberg \( L \)-functions

In this section, we deduce some facts we will need on automorphic forms of Drinfeld-type and Rankin-Selberg \( L \)-functions from [CXY] §§3–4.

Let \( M(P_0) \) (resp. \( S(P_0) \)) be the space of automorphic forms (resp. cusp forms) of Drinfeld type for \( \Gamma_0(P_0) \) with Petersson inner product \( \langle f, f \rangle_{P_0} \). Our assumption that \( \deg(P_0) \geq 3 \) ensures that the dimension of the space \( S(P_0) \) is positive (see [23]). For each place \( v \) of \( k \), let \( T_v \) be the Hecke operator on \( M(P_0) \) corresponding to \( v \). Let \( f \in M(P_0) \) be a normalized Hecke eigenform, and let \( \lambda_v(f) \in \mathbb{R} \) be the Hecke eigenvalue corresponding to \( T_v \). Then, we have \( \lambda_v(f) = 1 \), and for each finite place \( v \) of \( k \), we have
- \( |\lambda_v(f)| \leq 2q_v^{1/2} \) if \( v \neq P_0 \),
- \( \lambda_{P_0}(f)^2 = 1 \),

where \( q_v \) is the cardinality of the residue field \( \mathbb{F}_v \) of \( k_v \). The first inequality is the Ramanujan bound, and the second inequality holds since \( f \) has trivial central character (under our assumptions, any form \( f \in M(P_0) \) has trivial central character \( w_f \), since \( w_f \) can be identified with a character on the ideal class group \( \text{Pic}(A) \), and the latter group is trivial; see the last paragraph of [CXY] §3.1).

Let \( f \in S(P_0) \) be a normalized newform and \( \chi \) be a character of \( \text{Pic}(\mathcal{O}_D) \). Define the Rankin-Selberg \( L \)-function
\[ L(f \times \chi, s) := \prod_v L_v(f \times \chi, s), \]
where the product is over all places \( v \) of \( k \), and the local factors are given by
\[ L_v(f \times \chi, s) := \prod_{1 \leq j, k \leq 2} (1 - \alpha_v^{(j)}(f)c_v^{(k)}(\chi)q_v^{-\frac{s+1}{2}})^{-1}, \]
where the numbers \( \alpha_v^{(j)}(f), c_v^{(k)}(\chi) \) are determined as follows:
Therefore, we can write
\[ m = L \]
where
\[ w \]
We have Theorem 4.1.

We adapt the argument in [AT, Thm. 3.3] used to bound the degree one

\[ \lambda \]
\[ v \]
Similarly, under our assumptions, the place \( v = \infty \) is ramified in \( K/k \), hence using that \( \lambda(\infty) = 1, \chi(K_{\infty}) = 1, \) and \( q_{\infty} = q \), we get
\[ L(1) = (1 - q^{-(s+\frac{1}{2})})^{-1}. \]

Similarly, under our assumptions, the place \( v = P_0 \) is inert in \( K/k \), hence using that \( \lambda(P_0) = 1, \chi(k) = 1, \) and \( q_{P_0} = q \), we get
\[ L(P_0,1) = (1 - \chi(w_{P_0})q_{P_0}^{2(s+\frac{1}{2})})^{-1}. \]

where \( w_{P_0} \mid P_0 \). Therefore, we have the Euler product
\[ L(f \times \chi, s) = (1 - q^{-(s+\frac{1}{2})})^{-1}(1 - \chi(w_{P_0})q_{P_0}^{2(s+\frac{1}{2})})^{-1} \]
\[ \times \prod_{v \mid P_0, v \text{ inert}} \prod_{1 \leq j,k \leq 2} (1 - \alpha_v^{(j)}(f)\alpha_v^{(k)}(\chi)q_v^{-(s+\frac{1}{2})})^{-1}. \]  

4. Bounding the central value \( L(f \times \chi, 1/2) \)

In this section, we will prove the following Lindelöf-type bound for the central value \( L(f \times \chi, 1/2) \).

**Theorem 4.1.** We have
\[ |L(f \times \chi, 1/2)| \leq \exp\left( \frac{3m}{2\log_q(m/2)} + O(\log(\log(\log(m/2) + 1))q^{1/2}m^{1/2}) \right) \]
where \( m \in \mathbb{Z}^+ \) satisfies \( m = O(\deg(P_0) + \deg(D)) \).

**Proof.** We adapt the argument in [AT, Thm. 3.3] used to bound the degree one \( L \)-function \( L(\chi, s) \) at \( s = 1/2 \).

By work of Deligne [De] and Drinfeld [Dr], the \( L \)-function \( L(f \times \chi, s) \) is a polynomial of degree \( m = O(\deg(P_0) + \deg(D)) \) in \( q^{-s} \) and has zeros only on the critical line \( \text{Re}(s) = 1/2 \). Therefore, we can write
\[ L(f \times \chi, s) = a \prod_{k=1}^{m} (1 - \alpha_k q^{1/2}) = a q^{-ms} \prod_{k=1}^{m} (q^s - \alpha_k q^{1/2}) \]
for complex numbers \( a \neq 0 \) and \( \alpha_k \) with \( |\alpha_k| = 1 \). Taking logarithmic derivatives yields
\[ \frac{L'}{L}(f \times \chi, s) = \log(q) \left( -m + \sum_{k=1}^{m} \frac{1}{1 - \alpha_k q^{1/2} - s} \right). \]
Define
\[ F(s) := \sum_{k=1}^{m} \text{Re} \left( \frac{1}{1 - \alpha_k q^{s/2}} - \frac{1}{2} \right) . \]

Then for \( s \in \mathbb{R} \), we have
\[ \frac{L'(f \times \chi, s)}{L(f \times \chi, s)} = \log(q) \left( -\frac{m}{2} + F(s) \right) . \]  

(12)

Let \( s_0 \in \mathbb{R} \) be such that \( 1/2 < s_0 < 1/2 + 1/\log(q) \). Then integrating from \( 1/2 \) to \( s_0 \) gives
\[ \log |L(f \times \chi, 1/2)| - \log |L(f \times \chi, s_0)| \]
\[ = \frac{m}{2} \log(q)(s_0 - \frac{1}{2}) - \log(q) \sum_{k=1}^{m} \int_{1/2}^{s_0} \text{Re} \left( \frac{1}{1 - \alpha_k q^{x/2}} - \frac{1}{2} \right) dx . \]  

(13)

To estimate the second term on the RHS of (13), we use the following lemma (see [AT, Lem. 3.1]).

**Lemma 4.2.** Let \( \theta \) and \( 0 < t < 1 \) be real numbers. Then
\[ \int_{0}^{t} \text{Re} \left( \frac{1}{1 - e^{-x - i\theta}} - \frac{1}{2} \right) dx \geq 2 \cdot \frac{1 + e^{-t}}{1 - e^{-t}} \cdot \text{Re} \left( \frac{1}{1 - e^{-t - i\theta}} - \frac{1}{2} \right) . \]

Since \( |\alpha_k| = 1 \), we can write
\[ \alpha_k q^{x/2} = e^{-\log(q)(s - 1/2) + i\theta_k} \]
for some \( \theta_k \in \mathbb{R} \). Make the change of variables \( x = \log(q)(s - 1/2) \) to get
\[ \int_{1/2}^{s_0} \text{Re} \left( \frac{1}{1 - \alpha_k q^{x/2}} - \frac{1}{2} \right) dx = \frac{1}{\log(q)} \int_{0}^{\log(q)(s_0 - \frac{1}{2})} \text{Re} \left( \frac{1}{1 - e^{-x - i\theta_k}} - \frac{1}{2} \right) dx . \]

By Lemma 4.2 we have
\[ \int_{0}^{\log(q)(s_0 - \frac{1}{2})} \text{Re} \left( \frac{1}{1 - e^{-x - i\theta_k}} - \frac{1}{2} \right) dx \geq 2 \cdot \frac{1 + q^{s_0 - 1/2}}{1 - q^{s_0 - 1/2}} \cdot \text{Re} \left( \frac{1}{1 - \alpha_k q^{s_0 - 1/2}} - \frac{1}{2} \right) . \]

Then applying this bound in (13) gives
\[ \log |L(f \times \chi, 1/2)| - \log |L(f \times \chi, s_0)| \leq \frac{m}{2} \log(q)(s_0 - \frac{1}{2}) - 2 \cdot \frac{1 + q^{s_0 - 1/2}}{1 - q^{s_0 - 1/2}} \cdot F(s_0) . \]  

(14)

Define \( h := \lfloor \log_q(m/2) \rfloor \). For \( \text{Re}(s) > 0 \), the integral
\[ \frac{1}{2\pi i} \int_{2 + \frac{1}{2\log(q)}}^{2 + \frac{2h + 1}{\log(q)}} \frac{L'(f \times \chi, s + w) q^{hw} q^{-w}}{(1 - q^{-w})^2} dw \]
can be computed in two different ways, first by expanding
\[ \frac{L'(f \times \chi, s)}{L(f \times \chi, s)} = \sum_{n=1}^{\infty} \frac{c_{f,K}(n)}{n^s} . \]
and integrating term by term, and second by systematically continuing to the left and picking up the residues at the poles. There is a double pole at \( w = 0 \), and simple poles at the values of \( w \) for which \( q^{s+w} = \alpha_k q^{1/2} \). This yields the identity

\[
- \log(q)^2 \sum_{n=1}^{h} c_{f,K}(n) \log(q^{h-n}) q^{ns} = h \log(q)^{-1} \frac{L'}{L}(f \times \chi, s) + \log(q)^{-2} \left( \frac{L'}{L}(f \times \chi, s) \right)'
+ \sum_{k=1}^{m} \frac{(\alpha_k q^{1/2-s})^h \alpha_k^{-1} q^{s-1/2}}{(1 - \alpha_k^{-1} q^{s-1/2})^2}.
\]

Integrate from \( s_0 \) to \( \infty \), take real parts, and multiply by \( \log(q)^2 \) to obtain the identity

\[
h \log(q) \log |L(f \times \chi, s_0)| = - \frac{L'}{L}(f \times \chi, s_0) + \log(q)^{-1} \sum_{n=1}^{h} c_{f,K}(n) \log(q^{h-n}) q^{ns_0}
+ \log(q)^2 \sum_{k=1}^{m} \int_{s_0}^{\infty} \Re \left( \frac{(\alpha_k q^{1/2-s})^h \alpha_k^{-1} q^{s-1/2}}{(1 - \alpha_k^{-1} q^{s-1/2})^2} \right) ds.
\]

To estimate the third term on the RHS of (15), we use the following lemma (see [AT, Lem. 3.2]).

**Lemma 4.3.** For \( s > s_0 > 1/2 \) we have

\[
\left| \frac{\alpha_k^{-1} q^{s-1/2}}{(1 - \alpha_k^{-1} q^{s-1/2})^2} \right| \leq \log(q)^{-1} \left( s_0 - \frac{1}{2} \right)^{-1} \Re \left( \frac{1}{1 - \alpha_k q^{1/2-s_0}} \right).
\]

By Lemma 4.3 we have

\[
\int_{s_0}^{\infty} \Re \left( \frac{(\alpha_k q^{1/2-s})^h \alpha_k^{-1} q^{s-1/2}}{(1 - \alpha_k^{-1} q^{s-1/2})^2} \right) ds \leq h^{-1} \log(q)^{-2} (s_0 - \frac{1}{2})^{-1} \Re \left( \frac{1}{1 - \alpha_k q^{1/2-s_0}} \right) q^{h(1/2-s_0)}.
\]

Then applying this bound in (15) gives

\[
\log |L(f \times \chi, s_0)| \leq -h^{-1} \log(q)^{-1} \frac{L'}{L}(f \times \chi, s_0) + h^{-1} \log(q)^{-2} \sum_{n=1}^{h} c_{f,K}(n) \log(q^{h-n}) q^{ns_0}
+ h^{-2} \log(q)^{-1} \left( s_0 - \frac{1}{2} \right)^{-1} \left( F(s_0) + \frac{m}{2} \right) q^{h(1/2-s_0)}.
\]

We now apply (16) in (14), then use (12) to get

\[
\log |L(f \times \chi, 1/2)| \leq \frac{m}{2} \left( \log(q)(s_0 - \frac{1}{2}) + h^{-1} + h^{-2} \log(q)^{-1} (s_0 - \frac{1}{2})^{-1} q^{h(1/2-s_0)} \right)
+ F(s_0) \left( (s_0 - \frac{1}{2})^{-1} \frac{q^{h(1/2-s_0)}}{h^2 \log(q)} - 2 \cdot \frac{1 + q^{1/2-s_0}}{1 - q^{1/2-s_0}} - h^{-1} \right)
+ h^{-1} \log(q)^{-2} \sum_{n=1}^{h} c_{f,K}(n) \log(q^{h-n}) q^{ns_0}.
\]

Choose \( s_0 = 1/2 + 1/h \log(q) \). Then

\[
h^{-2} \log(q)^{-1} (s_0 - \frac{1}{2})^{-1} q^{h(1/2-s_0)} < h^{-1}
\]
and
\[
(s_0 - \frac{1}{2})^{-1} \cdot \frac{q^{h(\frac{1}{2} - s_0)}}{h^2 \log(q)} - 2 \cdot \frac{1 + q^{\frac{1}{2} - s_0}}{1 - q^{\frac{1}{2} - s_0}} - h^{-1} < 0.
\]
Since \(F(s_0) > 0\), it follows that
\[
\log |L(f \times \chi, 1/2)| \leq \frac{3m}{2h} + h^{-1} \log(q)^{-2} \sum_{n=1}^{h} \frac{c_{f,K}(n) \log(q^{h-n})}{n q^{n s_0}}.
\] (17)

We next bound the coefficients \(c_{f,K}(n)\). Taking the logarithmic derivative of the Euler product \(L(f \times \chi, s)\), yields
\[
\frac{L'(f \times \chi, s)}{L(f \times \chi, s)} = -\log(q) \frac{q^{-(s+\frac{1}{2})}}{1 - q^{-(s+\frac{1}{2})}} - 2 \log(q_{P_0}) \frac{\chi(w_{P_0}) q_{P_0}^{-2(s+\frac{1}{2})}}{1 - \chi(w_{P_0}) q_{P_0}^{-2(s+\frac{1}{2})}} - \sum_{v \mid P_0} \sum_{1 \leq j, k \leq 2} \frac{\alpha_v^{(j)}(f) c_v^{(k)}(\chi) q_v^{-(s+\frac{1}{2})}}{1 - \alpha_v^{(j)}(f) c_v^{(k)}(\chi) q_v^{-(s+\frac{1}{2})}}.
\]

Since the eigenvalues \(\lambda_v(f)\) are real, the two complex conjugate roots \(\alpha_v^{(j)}(f)\) of \(X^2 - \lambda_v(f) X + q_v\) have modulus \(|\alpha_v^{(j)}(f)| = q_v^{1/2}\). Hence
\[
|\alpha_v^{(j)}(f) c_v^{(k)}(\chi) q_v^{-(s+\frac{1}{2})}| \leq 1,
\] (18)
so that (say, for \(\text{Re}(s) > 0\))
\[
\frac{L'(f \times \chi, s)}{L(f \times \chi, s)} = -\log(q) \sum_{n=1}^{\infty} (q^{-(s+\frac{1}{2})})^n - 2 \log(q_{P_0}) \sum_{n=1}^{\infty} (\chi(w_{P_0}) q_{P_0}^{-2(s+\frac{1}{2})})^n - \sum_{v \mid P_0} \sum_{1 \leq j, k \leq 2} \sum_{n=1}^{\infty} (\alpha_v^{(j)}(f) c_v^{(k)}(\chi) q_v^{-(s+\frac{1}{2})})^n.
\]

Moreover, since \(q_v = q^{\deg(v)}\) where \(\deg(v)\) is the degree of the residue field \(\mathbb{F}_v\) of \(k_v\), a calculation yields
\[
\frac{L'(f \times \chi, s)}{L(f \times \chi, s)} = \sum_{n=1}^{\infty} \frac{a(n)}{q^{ns}} + \sum_{n=1}^{\infty} \frac{b(n)}{q^{ns}} + \sum_{n=1}^{\infty} \frac{c(n)}{q^{ns}},
\]
where
\[
a(n) := -\log(q) q^{-n/2},
\]
\[
b(n) := \begin{cases} -2 \deg(P_0) \log(q)(\chi(w_{P_0}) q_{P_0}^{-1})^{n/2 \deg(P_0)}, & \text{if } 2 \deg(P_0) \mid n \\ 0, & \text{if } 2 \deg(P_0) \nmid n \end{cases}
\]
and
\[
c(n) := -\log(q) \sum_{d \mid n} d \sum_{v \mid P_0 \infty} \sum_{\deg(v) = d} (\alpha_v^{(j)}(f) c_v^{(k)}(\chi) q_v^{-(s+\frac{1}{2})})^{n/d}.
\]
Since \( c_{f,K}(n) = a(n) + b(n) + c(n) \), then using (18) we estimate
\[
|c_{f,K}(n)| \leq |a(n)| + |b(n)| + |c(n)| 
\leq \log(q) + 2 \log(q) \deg(P_0) + 4 \log(q) \sum_{d|n} d \cdot \# \{ v : P_0 \in \deg(v) = d \} 
\leq \log(q) + 2 \log(q) \deg(P_0) + 4 \log(q) \sum_{d|n} dq^d 
\leq \log(q)(1 + 2 \deg(P_0) + 4\sigma(n)q^n),
\]
where \( \sigma(n) := \sum_{d|n} d \).

We have
\[
\sigma(n) = O(\log(\log(n))n).
\]
Therefore, using that \( s_0 > 1/2 \), we apply these bounds in (17) to get
\[
\log |L(f \times \chi, 1/2)| \leq \frac{3m}{2h} + h^{-1} \log(q)^{-2} \sum_{n=1}^h c_{f,K}(n) \log(q^{h-n}) \frac{nq^{nso}}{nq^{nso}} 
\leq \frac{3m}{2h} + O(\log(\log(h))q^{h/2}).
\]
Since \( \log_q(m/2) \leq h \leq \log_q(m/2) + 1 \) (so that \( q^{h/2} \leq q^{1/2(m/2)^{1/2}} \)), we get
\[
|L(f \times \chi, 1/2)| \leq \exp\left(\frac{3m}{2\log_q(m/2)} + O(\log(\log_q(m/2) + 1))q^{1/2m^{1/2}}\right).
\]
This completes the proof. \( \square \)

5. The function field analog of Gross’s formula

In this section, we study an analog of Gross’s formula [Gr] over rational function fields due to Papikian [Pa], Wei and Yu [WY].

First, we recall some facts from the introduction. The Shimura curve \( X_{P_0} \), is the disjoint union of \( n \) genus zero curves \( X_i \) defined over \( k \). Hence if Pic\( (X_{P_0}) \) denotes the Picard group of \( X_{P_0} \) and if \( e_i \) denotes the class of degree 1 in Pic\( (X_{P_0}) \) corresponding to the component \( X_i \), we have
\[
\text{Pic}(X_{P_0}) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n.
\]
Since a Gross point \( x \in \text{Gr}_{D,P_0} \) lies on a component \( X_i \), it determines a class \( e_x \) in Pic\( (X_{P_0}) \) which for notational convenience we continue to denote by \( x \). We denote the action of Pic\( (\mathcal{O}_D) \) on \( \text{Gr}_{D,P_0} \) by \( x \mapsto x^\sigma \) for \( \sigma \in \text{Pic}(\mathcal{O}_D) \).

The Gross height pairing
\[
\langle , \rangle : \text{Pic}(X_{P_0}) \times \text{Pic}(X_{P_0}) \to \mathbb{Z}
\]
is defined on generators by \( \langle e_i, e_j \rangle = w_i \delta_{ij} \) and extended bi-additively to Pic\( (X_{P_0}) \), where \( w_i := \#(R_i^x)/(q - 1) \).

Also, recall that \( M(P_0) \) (resp. \( S(P_0) \)) denotes the space of automorphic forms (resp. cusp forms) of Drinfeld type for \( \Gamma_0(P_0) \) with the Petersson inner product \( \langle f, f \rangle_{P_0} \).

Let \( \mathcal{F}(P_0) \) be an orthogonal basis for \( S(P_0) \) consisting of normalized newforms. By the Jacquet-Langlands correspondence over \( k \) and the multiplicity-one theorem, for each
form \( f \in \mathcal{F}(P_0) \), there is a unique one-dimensional eigenspace \( \mathbb{R}e_f \) in \( \text{Pic}(X_{P_0}) \otimes \mathbb{R} \) such that \( \langle e_f, e_f \rangle = 1 \) and \( t_m e_f = \lambda_m(f) e_f \) for each monic polynomial \( m \in A \) with \( (m, P_0) = 1 \), where \( t_m \) is the Hecke correspondence and \( \lambda_m(f) \) is the eigenvalue for \( f \) associated to the Hecke operator \( T_m \) (see e.g. [W, §2.4, §4.4.1]).

Let \( e_f \in \text{Pic}(X_{P_0}) \otimes \mathbb{R} \) correspond to \( f \in \mathcal{F}(P_0) \) as above, and define

\[
e^* := \sum_{i=1}^{n} w_i e_i.
\]

Then an orthonormal basis for \( \text{Pic}(X_{P_0}) \otimes \mathbb{R} \) is given by

\[
\left\{ \frac{e^*}{\sqrt{\langle e^*, e^* \rangle}} \right\} \cup \{ e_f : f \in \mathcal{F}(P_0) \}.
\]

Given a character \( \chi \) of \( \text{Pic}(\mathcal{O}_D) \) and a Gross point \( x \in \text{Gr}_{D,P_0} \), define

\[
c_{\chi} := \sum_{\sigma \in \text{Pic}(\mathcal{O}_D)} \chi(\sigma) x^\sigma \in \text{Pic}(X_{P_0}) \otimes \mathbb{C}.
\]

Moreover, given a form \( f \in \mathcal{F}(P_0) \), recall that \( L(f \otimes \chi, s) \) denotes the Rankin-Selberg \( L \)-function associated to \( f \) and \( \chi \) (normalized so that the central value occurs at \( s = 1/2 \)). Then Papikian [Pa], Wei and Yu [WY, Thm. 3.3] proved the following Gross-type formula (recall that \( D \) is irreducible and \( \deg(D) \) is odd)

\[
L(f \otimes \chi, 1/2) = \frac{\langle f, f \rangle_{P_0}}{q^{\deg(D)/2}} \left| \langle c_{\chi}, e_f \rangle \right|^2,
\]

where \( \langle f, f \rangle_{P_0} \) is the Petterson inner product.

We now give an alternative description of (19) which will be useful for our calculations. Let \( M^e_B(P_0) \) be the vector space of \( \mathbb{C} \)-valued functions on \( \text{Pic}(X_{P_0}) \otimes \mathbb{Z} \mathbb{C} \), with inner product

\[
\langle \phi, \psi \rangle := \sum_{i=1}^{n} w_i \phi(e_i) \overline{\psi(e_i)}.
\]

Then the map which sends a generator \( e_i \) to its characteristic function \( \delta_{e_i} \) induces an isomorphism

\[
\text{Pic}(X_{P_0}) \otimes \mathbb{Z} \mathbb{C} \cong M^e_B(P_0)
\]

defined by

\[
e = \sum_{i=1}^{n} c_i e_i \mapsto \bar{e} := \sum_{i=1}^{n} c_i \delta_{e_i}.
\]

Moreover, this map is an isometry of inner-product spaces, i.e., \( \langle \bar{e}, \bar{e}' \rangle = \langle e, e' \rangle \) for any \( e, e' \in \text{Pic}(X_{P_0}) \otimes \mathbb{Z} \mathbb{C} \).

Let \( \tilde{f} = \bar{e_f} \) denote the image of \( e_f \) under this isomorphism. Then an orthonormal basis for \( M^e_B(P_0) \) is given by

\[
\left\{ \frac{\bar{e}}{\sqrt{\langle \bar{e}, \bar{e} \rangle}} \right\} \cup \{ \tilde{f} : f \in \mathcal{F}(P_0) \}.
\]
We can now express [19] as

$$L(f \otimes \chi, 1/2) = \frac{\langle f, \tilde{f} \rangle_{P_0}}{q^{\deg(D)+1}} \sum_{\sigma \in \Pic(O_D)} \frac{\chi(\sigma) f(\sigma)}{w_{\sigma}}$$

where we write $w_{\sigma}$ for $w_i$ if $\sigma$ lies in the class $e_i$.

6. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We begin by showing that

$$\frac{N_{G_D, P_0, e_i}}{|G_D|} = \mu_{P_0}(e_i) + \frac{1}{w_i} \frac{1}{h(D)} \sum_{\chi \in \Pic(O_D)} \sum_{f \in \mathcal{F}(P_0)} \langle \tilde{e}_i, \tilde{f} \rangle W_{\chi, D, \tilde{f}},$$

where the Weyl sum is defined by

$$W_{\chi, D, \tilde{f}} := \sum_{\sigma \in \Pic(O_D)} \chi(\sigma) w_{\sigma} f(\sigma).$$

We have

$$w_i N_{G_D, P_0, e_i} = w_i \# \{ \sigma \in G_D : x_{0, D}^{\sigma} = e_i \} = \sum_{\sigma \in G_D} w_{\sigma} \tilde{e}_i(\sigma).$$

By decomposing the function $\tilde{e}_i$ into a Hecke basis in $M^\mathbb{R}_{G_D}(P_0)$, we get

$$\tilde{e}_i(z) = \frac{\langle \tilde{e}_i, \tilde{e}^* \rangle}{\langle \tilde{e}^*, \tilde{e}^* \rangle} \tilde{e}^*(z) + \sum_{f \in \mathcal{F}(P_0)} \langle \tilde{e}_i, \tilde{f} \rangle \sum_{\sigma \in G_D} w_{\sigma} f(\sigma).$$

Therefore

$$w_i N_{G_D, P_0, e_i} = \frac{\langle \tilde{e}_i, \tilde{e}^* \rangle}{\langle \tilde{e}^*, \tilde{e}^* \rangle} \sum_{\sigma \in G_D} w_{\sigma} e^*(\sigma) + \sum_{f \in \mathcal{F}(P_0)} \langle \tilde{e}_i, \tilde{f} \rangle \sum_{\sigma \in G_D} w_{\sigma} f(\sigma).$$

Now, recall that the probability measure $\mu_{P_0}$ on $S$ is defined by

$$\mu_{P_0}(e_i) := \frac{w_i^{-1}}{\sum_{j=1}^n w_j^{-1}}$$

where $w_i := \#(R_i^\times)/(q-1)$. Then, using that $\langle \tilde{e}_i, \tilde{e}^* \rangle = 1$ for all $i$ and $\langle \tilde{e}^*, \tilde{e}^* \rangle = \sum_{j=1}^n w_j^{-1}$, we get

$$\frac{\langle \tilde{e}_i, \tilde{e}^* \rangle}{\langle \tilde{e}^*, \tilde{e}^* \rangle} = w_i \mu_{P_0}(e_i).$$

Also, we compute

$$\sum_{\sigma \in G_D} w_{\sigma} e^*(\sigma) = \sum_{\sigma \in G_D} w_{\sigma} \sum_{i=1}^n \frac{1}{w_i} \tilde{e}_i(\sigma) = |G_D|.$$

Lastly, by Fourier analysis we have

$$\frac{1}{|G_D|} \sum_{\sigma \in G_D} w_{\sigma} \tilde{f}(\sigma) = \frac{1}{h(D)} \sum_{\chi \in \Pic(O_D)} \sum_{\sigma \in \Pic(O_D)} \chi(\sigma) w_{\sigma} \tilde{f}(\sigma).$$

Then combining these calculations yields [21].
We now turn to the proof of Theorem 1.3. By Cauchy’s inequality, we have
\[
\left| \frac{N_{G_D, P_0, e_i}}{|G_D|} - \mu_{P_0}(e_i) \right| \leq \frac{1}{w_i h(D)} \sum_{\chi \in \text{Pic}(O_D), \chi|G_D} |\chi|^2 M_1^{1/2} M_2^{1/2}, \tag{22}
\]
where
\[
M_1 := \sum_{f \in F(P_0)} |\langle \tilde{e}_i, \tilde{f} \rangle|^2, \quad M_2 := \sum_{f \in F(P_0)} |W_{\chi, D, \tilde{f}}|^2.
\]
By Bessel’s inequality, we have
\[
M_1 \leq \langle \tilde{e}_i, \tilde{e}_i \rangle = w_i.
\]
Also, by the period formula (20) we get
\[
|W_{\chi, D, \tilde{f}}|^2 = q^{1/2} \|D\|^{1/2} L(f \otimes \chi, 1/2),
\]
which yields
\[
M_2 = q^{1/2} \|D\|^{1/2} \sum_{f \in F(P_0)} \frac{L(f \otimes \chi, 1/2)}{\langle f, f \rangle_{P_0}}.
\]

Theorem 4.1 gives the uniform Lindelöf bound
\[
L(f \otimes \chi, 1/2) \ll q^m \ll \|P_0\|^\varepsilon \|D\|^\varepsilon,
\]
where we recall that \(m = O(\deg(P_0) + \deg(D))\). Then applying this bound gives
\[
M_2^{1/2} \ll \varepsilon C(P_0)^{1/2} q^{1/4} \|P_0\|^\varepsilon/2 \|D\|^{1/4+\varepsilon/2},
\]
where
\[
C(P_0) := \sum_{f \in F(P_0)} \frac{1}{\langle f, f \rangle_{P_0}}.
\]
We have \(\langle f, f \rangle_{P_0} \gg 1\) (see for example [Ge3, §3]). Moreover, since the dimension of \(S(P_0)\) is equal to \(h_{P_0} - 1\), where
\[
h_{P_0} := \frac{1}{q^2 - 1} (\|P_0\| - 1) + \frac{q}{2(q + 1)} \left(1 - (-1)^{\deg(P_0)}\right) \tag{23}
\]
(see for example [WY, p. 740]), we get
\[
\#F(P_0) \ll \|P_0\|. \tag{24}
\]
Hence
\[
C(P_0) \ll \|P_0\|.
\]
It now follows from (22) and the lower bound
\[
h(D) \gg \varepsilon q^{-1} \|D\|^{1/2-\varepsilon}
\]
that
\[
\left| \frac{N_{G_D, P_0, e_i}}{|G_D|} - \mu_{P_0}(e_i) \right| \ll \varepsilon \text{[Pic}(O_D) : G_D] q^{5/4} \|P_0\|^{1/2+3\varepsilon/2} \|D\|^{-1/4+3\varepsilon/2}.
\]
Replacing \(\varepsilon\) with \(2\varepsilon/3\), we complete the proof. \(\square\)
Remark 6.1. In analogy with the number field case, we should have
\[ \langle f, f \rangle_{P_0} \asymp \|P_0\| L(\text{Sym}^2(f), 1), \]
where the symmetric-square $L$-function $L(\text{Sym}^2(f), 1)$ satisfies the following Hoffstein-Lockhart-type bound at $s = 1$,
\[ L(\text{Sym}^2(f), 1) \gg \varepsilon \|P_0\|^{-\varepsilon}. \]
It then follows from (24) that
\[ C(P_0) \ll \|P_0\|^\varepsilon, \]
and we could replace $\|P_0\|^{1/2+\varepsilon}$ with $\|P_0\|^\varepsilon$ in Theorem 1.3, Corollary 1.4, and $\|P_0\|^{6+\varepsilon}$ with $\|P_0\|^{4+\varepsilon}$ in Theorem 1.6.

References


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