1. Let \( f : \mathbb{R} \to \mathbb{R} \) be the function defined by \( f(x) = x^2 - 4 \). Express the following sets as intervals, unions of intervals, or finite sets. No formal proof is required.

(a) \( f([-2,0)) \)
(b) \( f(\mathbb{R}) \)
(c) \( f^{-1}(\{5\}) \)
(d) \( f^{-1}([0,5]) \)

Solution: (a) The graph of \( y = x^2 - 4 \) is a parabola pointing in the positive \( y \)-direction and with vertex at \((x, y) = (0, -4)\). We see that \( f(-2) = 0 \) and \( f(0) = -4 \), and so the image of \([-2, 0)\) will contain the points in between, namely \( f([-2,0)) = (-4, 0] \).

(b) By the same reasoning as in (a), the smallest value that \( f \) attains is \(-4\) at \( x = 0 \), and so \( f(\mathbb{R}) = [-4, \infty) \).

(c) The set \( f^{-1}(5) \) consists of all \( x \in \mathbb{R} \) such that \( f(x) = x^2 - 4 = 5 \). We see that this occurs when \( x = \pm 3 \), and so \( f^{-1}(5) = \{-3, 3\} \).

(d) Here we want to find all \( x \in \mathbb{R} \) such that \( 0 \leq f(x) \leq 5 \). After some calculation we see that \( f^{-1}([0,5]) = [-3, -2] \cup [2, 3] \).

2. Determine if the given function is injective, surjective, both, or neither. No formal proof is required.

(a) \( f : \mathbb{R} \to [-1,1], \quad f(x) = \cos^2(x) \)
(b) \( g : \mathbb{R}^* \to \mathbb{R}^*, \quad g(x) = \frac{1}{x} \quad (\mathbb{R}^* = \mathbb{R} - \{0\}) \)
(c) \( h : \mathbb{Z} \to 2\mathbb{Z}, \quad h(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases} \)
(d) \( j : \mathbb{Z} \to \mathbb{Z}, \quad j(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases} \)

Solution: (a) \( f \) is not one-to-one, since for example \( f(0) = f(2\pi) = 1 \). Also, \( f \) is not surjective, since \( \cos^2(x) \geq 0 \) for all \( x \in \mathbb{R} \). Therefore \( f \) is neither injective nor surjective.

(b) \( g \) is both injective and surjective. If \( f(x_1) = f(x_2) \), then this means that \( 1/x_1 = 1/x_2 \), from which we see that \( x_1 = x_2 \). Therefore \( f \) is injective. If \( y \in \mathbb{R}^* \), then \( f(1/y) = y \), and so \( f \) is surjective.

(c) \( h \) is surjective but not injective. Since \( h(0) = h(1) = 0 \), we see that \( h \) is not injective. If \( m \in 2\mathbb{Z} \) is some even number, then \( m + 1 \) is odd and then \( h(m+1) = (m+1) - 1 = m \). Therefore \( h \) is surjective.

(d) \( j \) is injective but not surjective. From the definition we see that if \( n \) is even, then \( j(n) = n+1 \) is odd, and if \( n \) is odd, then \( j(n) = 2n \), which is odd. So then we can show that \( j(n) \neq 4 \) for any value of \( n \). If \( j(n) \) were to be 4, then \( n \) would have to be odd and we would have to have \( 2n = 4 \). But \( 2n = 4 \) implies that \( n = 2 \), which is not odd. Therefore
$j$ is not surjective. To see that $j$ is one-to-one, we suppose that $j(n_1) = j(n_2)$. From the previous description, this implies that $n_1$ and $n_2$ are either both even or both odd. If both are even, then this implies that $n_1 + 1 = n_2 + 1$, and if both are odd, then $2n_1 = 2n_2$. In either case, we see that $n_1 = n_2$.

3. For each of the following statements, determine if it is true or false.

(a) If $f : A \to B$ is injective and $g : B \to C$ is injective, then $g \circ f : A \to C$ is injective.
(b) Let $\sim$ be the relation on $\mathbb{R}$ defined for $x, y \in \mathbb{R}$ by $x \sim y \iff |x - y| \leq 1$. Then $\sim$ is an equivalence relation.
(c) $2013 \equiv 7 \pmod{5}$.
(d) If $g : A \to A$ is a permutation of $A$, then $g \circ g$ is a permutation of $A$.

**Solution:** (a) **True** This was a result from class and the book (problem 7 in 3.3).

(b) **False** We see that $1 \sim 0$ and $0 \sim -1$, but $1 \not\sim -1$.

(c) **False** $2013 - 7 = 2006$, which is not divisible by 5.

(d) **True** The composition of two bijections is a bijection. A permutation $g : A \to A$ is a bijection from $A$ to $A$ by definition, so therefore $g \circ g$ is also a bijection (and permutation).

4. Let $g : [0, \infty) \to [0, \infty)$ be defined by $g(x) = x^2 + x$. Show that $g$ is invertible and find its inverse. Explain your reasoning.

**Solution:** First we show that $g$ is surjective. Since $g(0) = 0$ and $\lim_{x \to \infty} g(x) = \infty$, we see that $g$ takes on all values in $[0, \infty)$ by the intermediate value theorem. Next we show that $g$ is injective. The easiest way to see this is that $g'(x) = 2x + 1$, and so $g'(x) > 0$ for all $x \in (0, \infty)$. Therefore $g$ is strictly increasing on $[0, \infty)$ and therefore injective. To find the inverse of $g$, we let $x \in [0, \infty)$ and look for $y \in [0, \infty)$ such that $g(y) = x$. That is, we want to solve

$$y^2 + y = x$$

for $y$. This turns into the quadratic equation $y^2 + y - x = 0$. By the quadratic formula,

$$y = \frac{-1 \pm \sqrt{1 + 4x}}{2},$$

but only the ‘+’ solution will actually land in $[0, \infty)$ (the other will be negative). Therefore,

$$g^{-1}(x) = \frac{-1 + \sqrt{1 + 4x}}{2}.$$
5. Let $N = [0, \infty)$. Define a binary operation $\ast$ on $N$ by

$$a \ast b = (\sqrt{a} + \sqrt{b})^2, \quad a, b \in N.$$ 

Answer the following questions, and justify your answers.
(a) Verify that $1 \ast (1 \ast 4) = (1 \ast 1) \ast 4$. Does this imply that $\ast$ is associative?
(b) Is $\ast$ commutative? Why or why not?
(c) Show that $N$ contains an identity element with respect to $\ast$.
(d) Which elements of $N$ have inverses with respect to $\ast$?
(e) Let $S = \{m^2 \mid m \in \mathbb{Z}^+\} \subseteq N$. Show that $S$ is closed under $\ast$.

**Solution:**
(a) Quick calculations yield $1 \ast (1 \ast 4) = 1 \ast 9 = 16$ and $(1 \ast 1) \ast 4 = 4 \ast 4 = 16$. However, this does not imply that $\ast$ is associative, since in order to do we would need to check $a \ast (b \ast c) = (a \ast b) \ast c$ for general $a, b, c \in N$.
(b) Yes, $\ast$ is commutative. The reason is that

$$a \ast b = (\sqrt{a} + \sqrt{b})^2 = (\sqrt{b} + \sqrt{a})^2 = b \ast a,$$

for all $a, b \in N$.
(c) $0$ is an identity element of $N$ with respect to $\ast$. We check that for all $a \in N$,

$$a \ast 0 = (\sqrt{a} + \sqrt{0})^2 = (\sqrt{a})^2 = a.$$ 

Also $0 \ast a = a$, since $\ast$ is commutative.
(d) $0$ is the only element of $N$ with an inverse with respect to $\ast$. We see that $0$ is its own inverse: $0 \ast 0 = 0$. If $a \in N$, $a > 0$, then to find an inverse we would need to find $b \in N$ so that

$$a \ast b = (\sqrt{a} + \sqrt{b})^2 = 0.$$

However, the term in the middle is always positive if $a > 0$, so no such $b \in N$ can exist.
(e) Let $m^2, n^2 \in S$ with $m, n \in \mathbb{Z}^+$. Then

$$m^2 \ast n^2 = (\sqrt{m^2} + \sqrt{n^2})^2 = (m + n)^2.$$ 

Now $m + n \in \mathbb{Z}^+$, and so $(m + n)^2 \in S$. Therefore $S$ is closed under $\ast$.

6. Let $f : A \to B$ be a function. We define a relation $\approx$ on $A$ in the following way: for all $a_1$, $a_2 \in A$, we have $a_1 \approx a_2 \iff f(a_1) = f(a_2)$.
(a) Prove that $\approx$ is an equivalence relation on $A$.
(b) For $a \in A$, we let $[a]$ denote the equivalence class of $a$ with respect to $\approx$. Prove that $[a] = f^{-1}(\{f(a)\})$. (Note that here $f^{-1}$ indicates the inverse image, not an inverse function.)

**Solution:**
(a) We need to prove that $\approx$ is reflexive, symmetric, and transitive.

$\approx$ is reflexive: For $a \in A$, since $f(a) = f(a)$, we conclude that $a \approx a$.

$\approx$ is symmetric: Suppose that $a_1 \approx a_2$ for $a_1, a_2 \in A$. This means that $f(a_1) = f(a_2)$, which of course implies that $f(a_2) = f(a_1)$. Therefore, by the definition of $\approx$, we conclude that $a_2 \approx a_1$. 


Suppose that \( a_1 \approx a_2 \) and \( a_2 \approx a_3 \) for \( a_1, a_2, a_3 \in A \). These imply that \( f(a_1) = f(a_2) \) and \( f(a_2) = f(a_3) \). Therefore \( f(a_1) = f(a_3) \), from which we conclude that \( a_1 \approx a_3 \).

(b) From the definition of equivalence class we see that

\[
[a] = \{ x \in A \mid x \approx a \} = \{ x \in A \mid f(x) = f(a) \}.
\]

On the other hand,

\[
f^{-1}(\{f(a)\}) = \{ x \in A \mid f(x) \in \{f(a)\} \}.
\]

The condition that \( f(x) \in \{f(a)\} \) is equivalent to \( f(x) = f(a) \), since \( \{f(a)\} \) is a set containing only one element, namely \( f(a) \). Thus,

\[
f^{-1}(\{f(a)\}) = \{ x \in A \mid f(x) = f(a) \},
\]

and therefore comparing the descriptions of both sets, \([a] = f^{-1}(\{f(a)\})\).