Section 1.1 - Linear Functions and Math Models

Lines: Four basic things to know

1. The slope of the line
2. The equation of the line
3. The \(x\)-intercept
4. The \(y\)-intercept

1. **Slope:** If \((x_1, y_1)\) and \((x_2, y_2)\) are two distinct points on a line \(L\), then the slope, \(m\), of \(L\) is given by:

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}
\]

1. Find the slope of the line that passes through the points \((1, 3)\) and \((4, 9)\)

\[
\text{Slope} = m = \frac{3 - 9}{1 - 4} = \frac{-6}{-3} = 2
\]

2. Given that the slope of line \(L\) is \(-6\) and the change in \(y\) is 12 \((y\) increases by 12\), what is the corresponding change in \(x\)\?

\[
\text{Slope} = m = \frac{\Delta y}{\Delta x} \Rightarrow -6 = \frac{12}{\Delta x} \Rightarrow -6\Delta x = 12 \Rightarrow \Delta x = -2 \quad \text{"x decreases by 2"}
\]

2. **Equation of the line**
   
   (Point-Slope Form) The equation of the line that passes through the point \((x_1, y_1)\) and has slope \(m\) is given by:
   
   \[
y - y_1 = m(x - x_1)
   \]

(Slope-Intercept Form) The equation of the line that has slope \(m\) and intercepts the \(y\)-axis at the point \((0, b)\) is given by:

\[
y = mx + b
\]
3. Find the equation of the line described in Example 1. Pts. $\left(x_1, y_1\right), \left(y_1, y_2\right)$

\[
\text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2}{3} = 2
\]

Point-Slope
\[
y - y_1 = m(x - x_1)
\]

\[
y - 3 = 2(x - 1)
\]

\[
y - 3 = 2x - 2
\]

\[
y = 2x + 1
\]

4. Find the slope of the line given in standard form by:

\[
5x - 15y = 30 \rightarrow \text{Solve for } y:
\]

\[
5x - 15y = 30
\]

\[
-5x
\]

\[
\Rightarrow -15y = -5x + 30
\]

\[
\Rightarrow y = \frac{1}{3}x - 2
\]

Slope is $M = \frac{1}{3}$ "Coefficient of x"

3. $x$-intercept: The $x$-intercept is the point $(x_1, 0)$, where the line crosses the $x$-axis.

4. $y$-intercept: The $y$-intercept is the point $(0, y_1)$, where the line crosses the $y$-axis.

5. Find the $x$ and $y$ intercepts of the line given in Example 4.

\[
5x - 15y = 30 \quad \Rightarrow \quad \text{x-int set } y = 0: \quad 5x - 15y = 30 \quad \Rightarrow \quad \text{y-int set } x = 0:
\]

\[
5x = 30 \rightarrow x = 6
\]

\[
\left(x\text{-int}: (6, 0)\right)
\]

\[
-15y = 30 \rightarrow y = -2
\]

\[
\left(y\text{-int}: (0, -2)\right)
\]

Equations of Special Lines:

Horizontal Line: $y = \text{constant}$

Vertical Line: $x = \text{constant}$

6. Find the equations of the vertical and Horizontal lines that pass through the point $(12, 16)$

Horizontal Line: $y = 16$

Vertical Line: $x = 12$
Definition: A function is a rule that assigns to each value of $x$ one and only one value of $y$. We refer to $x$ as the independent variable and to $y$ as the dependent variable. The set of all possible values that $x$ can assume is the Domain and the set of all possible values that $y$ can assume is the Range.

Definition: A Mathematical Model is a mathematical description (usually involving a function or an equation) of a real-world occurrence. The purpose of the model is to understand a phenomenon and perhaps to make predictions about future behavior.

We will concentrate on linear functions/models (can be represented graphically by a line)

Applications:

Linear Depreciation

7. In 2009 Texas A&M installed a new machine in one of its departments at a cost of $250,000. The machine is depreciated linearly over 10 years with a scrap value of $10,000.

(a) Find an expression for the machine’s book value ($V$) in terms of the number of years since 2009 ($t$).

Find Eqn. of the line's

\[
\text{Slope } m = \frac{250000 - 10000}{0 - 10} = -24000.
\]

Point-Slope

\[
V - 250000 = -24000(t - 0)
\]

\[
\Rightarrow V = 250000 - 24000t + 250000
\]

\[
\Rightarrow V = -24000t + 250000
\]

(b) Find the machine’s book value in 2013.

2013 means $t = 4$ “4 yrs from 2009”

\[
V(4) = -24000(4) + 250000 = $154000
\]

(c) Find the rate at which the machine is being depreciated.

\[
\text{Rate is } -24000 \text{ $/year}
\]

"Value decreases by $24000/year"
Linear Demand and Supply Curves

**Linear Demand Function:** A linear demand function will have a **negative** slope because as the price, \( p = D(x) \), goes **up** the quantity demanded, \( x \), goes **down**.

8. At a unit price of $410, the quantity demanded of a certain commodity is 73 pounds. If the unit price increases to $680, the quantity demanded decreases by 27 pounds. 
   
   \[ \Delta x = -27 \]  
   
   (a) Find the demand equation \( p = D(x) \) (assuming it is linear) where \( p \) is the unit price and \( x \) is the quantity demanded for this commodity in pounds.
   
   \[ \Delta x = -27 \Rightarrow x_2 - x_1 = -27 \Rightarrow x_2 - 73 = -27 \Rightarrow x_2 = 46 \]
   
   \( \text{Points: (73, 410), (46, 680)} \)
   
   \[ \text{Slope } m = \frac{680 - 410}{46 - 73} = -10 \]
   
   \[ \text{Point-Slope: } y - 410 = -10(x - 73) \]
   
   \[ \Rightarrow y = -10x + 1140 \]

(b) Sketch the demand curve.

(c) What quantity would be demanded if the commodity were free?

\[ y = 0 \]

\[ 0 = -10x + 1140 \Rightarrow \text{Solve for } x \]

\[ -1140 \]

\[ \Rightarrow -1140 = -10x \Rightarrow x = 114 \text{ pounds} \]
Linear Supply Function: A linear supply function will have a positive slope because as the price, \( p = S(x) \), goes up the quantity supplied, \( x \), goes up.

9. Producers will make 2000 cell phones available when the unit price is $100. At a unit price of $200, 6000 cell phones will be marketed.

(a) Find the supply equation \( p = S(x) \) (assuming it is linear) where \( p \) is the cell phone price and \( x \) is the quantity of cell phones supplied.

\[
\begin{align*}
p &\approx (2000, 100), (6000, 200) \\
\text{slope} : m &\approx \frac{200 - 100}{6000 - 2000} = 0.025 \\
\text{point-slope} : \\
y - 100 &\approx 0.025(x - 2000) \\
y &\approx 0.025x - 50 + 100 \implies y = 0.025x + 50
\end{align*}
\]

(b) What is the unit price when 3000 cell phones are supplied?

\[
y = 0.025(3000) + 50 \\
\implies y = $125
\]

Cost, Revenue, and Profit Functions

**Cost Function** (cost of manufacturing \( x \) units of a product)

\[
C(x) = c x + F
\]

where \( c \) is the cost per unit and \( F \) is the fixed cost.

**Revenue Function** (revenue realized from selling \( x \) units of the product)

\[
R(x) = s x
\]

where \( s \) is the selling price.

**Profit Function**

\[
P(x) = R(x) - C(x) \implies \text{Revenue} - \text{Cost} \\
P(x) = sx - (cx + F) \\
\implies P(x) = sx - cx - F
\]
10. A manufacturer has a monthly fixed cost of $100,000 and a production cost of $14 for each unit produced. The product sells for $20/unit.

(a) What is the cost function?

\[ C(x) = FC + VC = 14x + 100000 \]

(b) What is the revenue function?

\[ R(x) = p \times x = 20x \]

(c) What is the profit function?

\[ P(x) = R(x) - C(x) = 20x - 14x - 100000 = 6x - 100000 \]

(d) Compute the profit (loss) corresponding to production levels of 12,000 and 20,000 units.

\[ P(12000) = 6(12000) - 100000 = -28000 \rightarrow \text{Loss of } $28000 \]

\[ P(20000) = 6(20000) - 100000 = 20000 \rightarrow \text{Gain/Profit of } $20000 \]
11. Phil makes and sells rugs at his roadside stand. His monthly fixed cost for owning the stand is $1125. If he makes and sells 17 rugs, his total costs are $1278 and he brings in $578 in revenue. Find Phil’s monthly cost, revenue, and profit functions (assuming they are linear). Let \( x \) be the number of rugs made and sold each month.

Cost: \( F = 1125 \rightarrow (0,1125) \)

Find \( c \) (slope) use \((17,1278)\)

\[
\frac{1278-1125}{17-0} = 9
\]

\( \Rightarrow \quad C(x) = 9x + 1125 \)

Revenue: The pt. \((0,0)\) is always on revenue line.

Find \( s \) (slope) use \((17,578)\)

\[
\frac{578-0}{17-0} = 34
\]

\( \Rightarrow \quad R(x) = 34x \)

Profit:

\[
P(x) = R(x) - C(x)
\]

\[
= 34x - 9x - 1125
\]

\[
= 25x - 1125
\]