On Some Local Operator Space Properties

Zhong-Jin Ruan
University of Illinois at Urbana-Champaign

Brazos Analysis Seminar at TAMU
March 25-26, 2017
Operator spaces are natural noncommutative quantization of Banach spaces.

Many Banach space results (like duality, tensor products, Grothendieck approximation property, and etc.) have the natural operator space analogue.

However, some local properties of Banach spaces fail for operator spaces.
A Banach space is a complete normed space \((V/\mathbb{C}, \| \cdot \|)\).

For Banach spaces, we consider

**Norms and Bounded Linear Maps.**
Classical Theory

\[ \ell_\infty(I) \]

Banach Spaces

\[(V, \| \cdot \|) \hookrightarrow \ell_\infty(I)\]

Noncommutative Theory

\[ B(H) \]

Operator Spaces

\[(V, ??) \hookrightarrow B(H)\]

norm closed subspaces of \( B(H) \)?
An operator space $V$ is defined to be a norm closed subspace of some $B(H)$ together with a matricial norm obtained by identifying each

$$M_n(V) \subseteq M_n(B(H)) = B(H^n).$$

For operator spaces, we consider

Matricial Norms and Completely Bounded Linear Maps.
Completely Bounded Maps

Let $\varphi : V \to W$ be a bounded linear map. For each $n \in \mathbb{N}$, we can define a linear map

$$\varphi_n : M_n(V) \to M_n(W)$$

by letting

$$\varphi_n([v_{ij}]) = [\varphi(v_{ij})].$$

In general, we have

$$\|\varphi\| \leq \|\varphi_2\| \leq \cdots \leq \|\varphi_n\| \leq \cdots$$

The map $\varphi$ is called completely bounded if

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\} < \infty.$$ 

In general $\|\varphi\|_{cb} \neq \|\varphi\|$. Let $t$ be the transpose map on $M_n(\mathbb{C})$. Then

$$\|t\|_{cb} = n, \text{ but } \|t\| = 1.$$
**Theorem:** If \( \varphi : V \to W \subseteq C_b(\Omega) \) is a bounded linear map, then \( \varphi \) is completely bounded with

\[
\|\varphi\|_{cb} = \|\varphi\|.
\]
Arveson-Wittstock-Hahn-Banach Theorem

Let $V \subseteq W \subseteq B(H)$ be operator spaces.

\[
\begin{array}{c}
W \\
\uparrow \quad \downarrow \tilde{\varphi} \\
V \quad \xrightarrow{\varphi} \quad B(H)
\end{array}
\]

with $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$.

In particular, if $B(H) = \mathbb{C}$, we have $\|\varphi\|_{cb} = \|\varphi\|$. This, indeed, is a generalization of the classical Hahn-Banach theorem.
Dual Operator Spaces

Let $V$ be an operator space. Then the dual space

$$V^* = B(V, \mathbb{C}) = CB(V, \mathbb{C})$$

has a natural operator space matrix norm given by

$$M_n(V^*) = CB(V, M_n(\mathbb{C})).$$

We call $V^*$ the operator dual of $V$. 
Examples of Operator Spaces

- **C*-algebras** $A$
  
  - $A = C_0(\Omega)$ or $A = C_b(\Omega)$ for locally compact space

- Reduced group C*-algebras $C^*_\lambda(G)$, full group C*-algebras $C^*(G)$

- von Neumann algebras $M$, i.e. strong operator topology (resp. w.o.t, weak* topology) closed *-subalgebras of $B(H)$

- $L_\infty(X, \mu)$ for some measure space $(X, \mu)$

- Group von Neumann algebras $VN(G)$

- **Subspaces of C*-algebras**
More Examples

- $T(\ell_2(\mathbb{N})) = K(\ell_2(\mathbb{N}))^* = B(\ell_2(\mathbb{N}))^*$;

- $M(\Omega) = C_0(\Omega)^*$, operator dual of C*-algebras $A^*$;

- $L_1(X, \mu) = L_\infty(X, \mu)^*$, operator predual of von Neumann algebras $R^*$;

- Fourier algebra $A(G) = VN(G)^*$

- Fourier-Stieltjes algebra $B(G) = C^*(G)^*$
Operator Space Structure on $L_p$ spaces

- $L_p$-spaces $L_p(X, \mu)$

$$M_n(L_p(X, \mu)) = (M_n(L_\infty(X, \mu)), M_n(L_1(X, \mu)))_\frac{1}{p}.$$  

- Non-commutative $L_p$-spaces $L_p(R, \varphi)$,

$$M_n(L_p(R, \varphi)) = (M_n(R), M_n(R_{*}^{op}))_\frac{1}{p},$$  

where $R_{*}^{op}$ is the operator predual of the opposite von Neumann algebra $R^{op}$.  

Related Books


Local Properties
Local Property of Banach Spaces

It is known from the Hahn-Banach theorem that given a finite dimensional Banach space \( E \), there exists an isometric inclusion

\[
E \hookrightarrow \ell_{\infty}(\mathbb{N}).
\]

**Question:** Since \( E \) is finite dimensional, can we

“approximately embed” \( E \) into a finite dimensional \( \ell_{\infty}(n) \)

for some positive integer \( n \in \mathbb{N} \) ?
Finite Representability in $\{\ell_{\infty}(n)\}$

**Theorem:** Let $E$ be a f.d. Banach space. For any $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq \ell_{\infty}(n(\varepsilon))$ such that

$$E \overset{1+\varepsilon}{\cong} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\| \|T^{-1}\| < 1 + \varepsilon.$$

In this case, we say that

- Every f.d. Banach space $E$ is **representable** in $\{\ell_{\infty}(n)\}$; and
- Every Banach space $V$ is **finitely** representable in $\{\ell_{\infty}(n)\}$. 
**Proof:** Since $E^*$ is finite dim, the closed unit ball $E_1^*$ is norm compact and thus totally bounded. For arbitrary $1 > \varepsilon > 0$, there exists finitely many functionals $f_1, \cdots, f_n \in E_1^*$ such that for every $f \in E_1^*$, there exists some $f_j$ such that

$$\|f - f_j\| < \frac{\varepsilon}{1 + \varepsilon}.$$ 

Then we can define a linear contraction

$$T : x \in E \rightarrow (f_1(x), \cdots, f_n(x)) \in \ell_\infty(n).$$ 

For any $f \in E_1^*$, there exists $f_j$ (with $1 \leq j \leq n$) such that $\|f - f_j\| < \frac{\varepsilon}{1 + \varepsilon}$ and thus we get

$$\|T(x)\| \geq |f_j(x)| \geq |f(x)| - |f(x) - f_j(x)| \geq |f(x)| - \frac{\varepsilon \|x\|}{1 + \varepsilon}.$$ 

This shows that

$$\|T(x)\| \geq \|x\| - \frac{\varepsilon \|x\|}{1 + \varepsilon} = \frac{\|x\|}{1 + \varepsilon}.$$ 

Therefore, $\|T^{-1}\| < 1 + \varepsilon.$
Finite Representatibility of Operator Spaces in \( \{M_n\} \)

Correspondingly, we say that an operator space \( V \) is finitely representable in \( \{M_n\} \) if for every f.d. subspace \( E \) and \( \varepsilon > 0 \), there exist \( n(\varepsilon) \in \mathbb{N} \) and \( F \subseteq M_n(\varepsilon) \) such that

\[
E \overset{1+\varepsilon}{\cong}_{\text{cb}} F,
\]

i.e., there exists a linear isomorphism \( T : E \to F \) such that

\[
\|T\|_{\text{cb}} \|T^{-1}\|_{\text{cb}} < 1 + \varepsilon.
\]

It is natural to ask

whether every operator space is finitely representable in \( \{M_n\} \)?
The answer is no.

**Theorem [Pisier 1995]:** Let $\ell_1(n)$ be the operator dual of $\ell_\infty(n)$. If $T : \ell_1(n) \to F \subseteq M_k$ is a linear isomorphism, then for $n > 2$

$$\|T\|_{cb}\|T^{-1}\|_{cb} \geq \frac{n}{2} \sqrt{n - 1}.$$ 

Pisier’s result shows that operator spaces need not be finitely representable in $\{M_n\}$. 
Why:

The reason is that we do not have matricial norm compactness for the matricial closed unit ball of finite dimensional dual space $E^*$

More precisely, suppose $E$ is a finite dimensional operator space. Then for each $n \in \mathbb{N}$, the $n \times n$ matrix space $M_n(E^*)$ is finite dimensional and thus its closed unit ball $M_n(E^*)_1$ is norm compact. However, the matricial closed unit ball $\{M_n(E^*)_1\}$ is not compact anymore.

This topic on “operator space compactness” has been studied in the literature.
What does the Finite Representability in \( \{M_n\} \) correspond to?

Suppose \( V \subseteq B(H) \) is an operator space, finitely representable in \( \{M_n\} \). Then for each f.d. operator subspace \( E \subseteq V \) and \( \varepsilon > 0 \), we can find complete bounded maps

\[
T : E \to F \subseteq M_n(\varepsilon) \quad \text{and} \quad T^{-1} : F \to E \subseteq V \subseteq B(H)
\]

with \( \|T\|_{cb}\|T^{-1}\|_{cb} < 1 + \varepsilon \).

We can apply the Arveson-Wittstock-Hahn-Banach extension theorem to get complete bounded extensions

\[
\Phi_{(E,\varepsilon)} : V \to M_n(\varepsilon) \quad \text{and} \quad \Psi_{(E,\varepsilon)} : M_n(\varepsilon) \to B(H)
\]

of \( T \) and \( T^{-1} \) with \( \|\Phi_{(E,\varepsilon)}\|_{cb} = \|T\|_{cb} \) and \( \|\Psi_{(E,\varepsilon)}\|_{cb} = \|T^{-1}\|_{cb} \).

From this, we can get two nets (or two sequences if \( V \) separable) of such maps such that

\[
\|\Psi_{(E,\varepsilon)} \circ \Phi_{(E,\varepsilon)}(x) - x\| \to 0 \quad (x \in V)
\]
**Remark:** If we like, we can choose \( \{ \Phi(E, \varepsilon) \} \) and \( \{ \Psi(E, \varepsilon) \} \) to be complete contractions.

This shows that if an operator space \( V \) is finitely representable in \( \{ M_n \} \), then \( V \) is an exact operator space.

Actually, the converse is also true. We have

**Theorem:** An operator space \( V \) is finitely representable in \( \{ M_n \} \) if and only if \( V \) is an exact operator space.
**Exact Operator Spaces/C*-algebras**

**Kirchberg:** A $C^*$-algebra $A$ is **exact** if we have the short exact sequence

$$0 \to K(\ell_2) \bar{\otimes} A \to B(\ell_2) \bar{\otimes} A \to Q(\ell_2) \bar{\otimes} A \to 0,$$

where $Q(H) = B(\ell_2)/K(\ell_2)$.

**Pisier:** An operator space is **exact** (more precisely **1-exact**) if we have the short exact sequence

$$0 \to K(\ell_2) \bar{\otimes} V \to B(\ell_2) \bar{\otimes} V \to Q(\ell_2) \bar{\otimes} V \to 0,$$

and this induces an **isometric isomorphism**

$$B(\ell_2) \bar{\otimes} V/K(\ell_2) \bar{\otimes} V \cong Q(\ell_2)/K(\ell_2) \bar{\otimes} V.$$
**Nuclearity**

An operator space (or a C*-algebra) $V$ is **nuclear** if there exists two nets of completely contractive maps $S_\alpha : V \to M_{n(\alpha)}$ and $T_\alpha : M_{n(\alpha)} \to V$ such that

$$\|T_\alpha \circ S_\alpha(x) - x\| \to 0$$

for all $x \in V$.

**CCAP and CBAP**

An operator space $V$ has the **CBAP** (resp. **CCAP**) if there exists a net of completely bounded (resp. completely contractive) finite rank maps $T_\alpha : V \to V$ such that

$$\|T_\alpha(x) - x\| \to 0$$

for all $x \in V$. 
Grothendick's Approximation Property

A Banach space is said to have Grothendicks’ AP if there exists a net of bounded finite rank maps $T_{\alpha} : V \to V$ such that $T_{\alpha} \to id_V$ uniformly on compact subsets of $V$.

We note that a subset $K \subseteq V$ is compact if and only if there exists a sequence $(x_n) \in c_0(V)$ such that

$$K \subseteq \overline{\text{conv}\{x_n\}}_{\|\cdot\|} \subseteq V.$$ 

Therefore, $V$ has Grothendick’s AP if and only if there exists a net of finite rank bounded maps $T_{\alpha}$ on $V$ such that

$$\|(T_{\alpha}(x_n)) - (x_n)\|_{c_0(V)} \to 0$$

for all $(x_n) \in c_0(V)$. 
Operator Space Approximation Property

An operator space $V$ is said to have the operator space approximation property (or simply, OAP) if there exists a net of finite rank bounded maps $T_\alpha$ on $V$ such that

$$\| [T_\alpha(x_{ij})] - [x_{ij}] \|_{K_\infty(V)} \to 0$$

for all $[x_{ij}] \in K_\infty(V)$, where we let $K_\infty(V) = \bigcup_{n=1}^\infty M_n(V)$.

In this case, we say that $T_\alpha \to id_V$ in the stable point-norm topology.

We say that $V \subseteq B(H)$ has the strong OAP if we can replace $K_\infty(V) = K_\infty \otimes V$ by $B(\ell_2) \otimes V$. 
Examples of Exact Reduced Group C*-algebras

Let $G$ be a discrete group. For the reduced group C*-algebra $C^*_\lambda(G)$,

Nuclearity $\Rightarrow$ CBAP $\Rightarrow$ strong OAP $\Leftrightarrow$ OAP $\Rightarrow$ Exactness.

$C^*_\lambda(\mathbb{F}_n)$ $\quad C^*_\lambda(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}))$ $\quad C^*_\lambda(SL(3, \mathbb{Z}))$
Examples of Non-Exact Full Group $C^*$-algebras

First let us recall

**Theorem:** If $G$ is a residually finite discrete group, then $G$ is amenable if and only if the full group $C^*$-algebra $C^*(G')$ is nuclear/exact.

Therefore, for residually finite non-amenable groups, such as

$$G = \mathbb{F}_n, \quad \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}), \quad SL(3, \mathbb{Z}),$$

the full group $C^*$-algebras $C^*(G')$ are not exact.
Brown-Guentner’s Construction of $\ell_p$-C*-algebras $C^*_\ell(p)(G)$
Recently, Brown-Guntner introduced a new class of group C*-algebras $C_{\ell_p}^*(G)$ for all $1 \leq p < \infty$. Let us recall that an $\ell_p$-representation of a discrete group $G$ is a unitary representation $\pi : G \to B(H)$, for which there exists a dense subspace $H_0$ of $H$ such that the coefficient function $s \to \langle \pi(s)\xi, \eta \rangle$ is an element of $\ell_p(G)$ for every $\xi, \eta$ in the dense subspace $H_0$.

We can define a $C^*$-norm $\| \cdot \|_{\ell_p}$ on the group ring $\mathbb{C}[G]$ by

$$\|x\|_{\ell_p} = \sup \{\|\pi(x)\| : \pi \text{ is an } \ell_p\text{-representation of } G\}.$$  

We let $C_{\ell_p}^*(G)$ denotes the induced $\ell_p$-$C^*$-algebra. Since 

$$\ell_\infty(G) \supseteq \ell_p(G) \supseteq \ell_2(G) \supseteq \ell_q(G) \supseteq \ell_1(G)$$

and larger ideal space contains more coefficient functions, we have

$$\| \cdot \|_{\ell_\infty} \geq \| \cdot \|_{\ell_p} \geq \| \cdot \|_{\ell_2} \geq \| \cdot \|_{\ell_q} \geq \| \cdot \|_{\ell_1}$$

on $\mathbb{C}[G]$. This gives us the surjective $C^*$-quotients

$$C^*(G) = C_{\ell_\infty}^*(G) \to C_{\ell_p}^*(G) \to C_{\ell_2}^*(G') \to C_{\ell_q}^*(G') \to C_{\ell_1}^*(G).$$
**Theorem (Brown-Guntner):** For $q \in [1, 2]$, we have

\[ C_{\ell_1}(G) = C_{\ell_q}(G) = C_{\ell_2}(G) = C_\lambda(G). \]

Therefore, for non-amenable groups, we are mainly interested in the $\ell_p$-goup C*-algebras $C_{\ell_p}(G)$ for any $2 < p < \infty$.

**Theorem (Brown-Guntner):** There exists $p \in (2, \infty)$ such that

\[ C^*(\mathbb{F}_n) \not\cong C_{\ell_p}(\mathbb{F}_n) \not\cong C_{\ell_q}(\mathbb{F}_n). \]

**Theorem (Okayasu):** Let $n \geq 2$ be a positive integer. For any $2 < q < p < \infty$ the canonical C*-quotients

\[ C^*(\mathbb{F}_n) \not\cong C_{\ell_p}(\mathbb{F}_n) \not\cong C_{\ell_q}(\mathbb{F}_n) \not\cong C_{\ell_2}(\mathbb{F}_n) = C_\lambda(\mathbb{F}_n) \]

are proper. So for $2 < p < \infty$, $C_{\ell_p}(\mathbb{F}_n)$ are all exotic group C*-algebras.
Examples of Non-Exact Exotic Group C*-algebras

**Theorem (R-Wiersma):** For each $2 < p < \infty$, the exotic group C*-algebras $C^*_\ell p(F_n)$ is not exact.

**Theorem (Wiersma):** If $H$ is a subgroup of a discrete group $G$, then for $2 < p < \infty$, $C^*_\ell p(H)$ is *-isomorphic to a C*-subalgebra of $C^*_\ell p(G)$. So if $C^*_\ell p(H)$ is exotic and distinct, then so are $C^*_\ell p(G)$.

**Theorem (R-Wiersma):** Let $G$ be a discrete group containing $\mathbb{F}_2$ as a subgroup. For instance,

$$G = F_n, \quad \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}), \quad SL(3, \mathbb{Z}).$$

For each $2 < p < \infty$, the exotic group C*-algebra $C^*_\ell p(G)$ is not exact.
Finite Representability in $\{\ell_1(n)\}$

A Banach space $V$ is finitely representable in $\{\ell_1(n)\}$ if for every finite dimensional subspace $E \subseteq V$ and $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq \ell_1(n(\varepsilon))$ such that

$$E^{1+\varepsilon} \cong F,$$

i.e., there exists a linear isomorphism $T : E \rightarrow F$ such that

$$\|T\| \|T^{-1}\| < 1 + \varepsilon.$$

Theorem: A Banach space $V$ is finitely representable in $\{\ell_1(n)\}$ if and only if $V$ is isometric to a subspace of $L_1(\Omega, \mu)$ space for some measure space $(\Omega, \mu)$.
To see this theorem, let \((\Omega, \mu)\) be a measure space and let

\[ E = \text{span}\{f_1, \cdots, f_n\} \]

be a f.d. subspace of \(L_1(\Omega, \mu)\). For each \(\varepsilon > 0\) and \(j = 1, \cdots, n\), we can find a simple function \(\phi_j\) such that \(\|f_j - \phi_j\|_1 < \varepsilon\).

Notice that each simple function \(\phi_j\) is a finite linear combination of some characteristic functions \(\chi_{\Omega_j^i}\) on non-zero measurable sets \(\Omega_j^i\). Without loss of generality, we can assume that all of these \(\Omega_j^i\) are mutually disjoint. Then \(\{\chi_{\Omega_j^i} / \mu(\Omega_j^i)\}\) form a canonical basis for some \(\ell_1(n(\varepsilon))\) space. Then the map

\[ T : f_j \in E \to \phi_j \in \ell_1(n(\varepsilon)) \]

is the map satisfying \(\|T\|\|T^{-1}\| < 1 + \varepsilon\).

On the other hand, if \(V\) is finitely representable in \(\{\ell_1(n)\}\), then we can show that \(V\) is isometric to a subspace of an ultraproduct \(\prod_{U} \ell_1(n)\) of \(\ell_1(n)\) and thus \(V\) is a subspace of some \(L_1\) space.
Operator Space Finite Representability in \( \{T_n\} \)

An operator space is finitely representable in \( \{T_n\} \) if for every f.d. sub-space space \( E \) and \( \varepsilon > 0 \), there exist \( n(\varepsilon) \in \mathbb{N} \) and \( F \subseteq T_n(\varepsilon) \) such that

\[
E \overset{1+\varepsilon}{\approx} F,
\]

i.e., there exists a linear isomorphism \( T : E \rightarrow F \) such that

\[
\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon.
\]
**Theorem [Effros-Junge-R]**: An operator space $V$ is finitely representable in $\{T_n\}$ if and only if there is a completely isometric embedding

$$V \hookrightarrow \prod_{U} T_n(\alpha).$$

**Examples:**

- $C(X)^*, T_n, T(\ell_2)$ are finitely representable in $\{T_n\}$.
- The operator dual of nuclear $C^*$-algebras are finitely representable in $\{T_n\}$.
- If $G$ is residually finite, then $C^*_\lambda(G)^*$ is finitely representable in $\{T_n\}$. 
Relation with QWEP

A C*-algebra $A \subseteq B(H)$ has the (Lance’s) WEP if there exists a completely positive map $\Phi : B(H) \to A^{**}$ such that $\Phi(x) = x$ for all $x \in A$.

A C*-algebra $A$ has the QWEP if $A = B/J$ for some C*-algebra $B$ with the WEP.

**Theorem [Effros-Junge-R]:** Let $A$ be a C*-algebra. Then $A^\ast$ is finitely representable in $\{T_n\}$ if and only if $A$ has the QWEP.

Therefore, the finite representability property of $A^\ast$ is equivalent to

**Kirchberg ’s conjecture 1993:**

Every C*-algebra has QWEP.

**A. Connes’ conjecture 1976:**

Every finite von Neumann algebra with separable predual is *-isomorphic to a von Neumann subalgebra of the ultrapower of the hyperfinite II$_1$ factor.
Recall that for any $2 < q < p < \infty$ we have the canonical (proper) $C^*$-quotients
\[ C^*(\mathbb{F}_n) \nrightarrow C^*_{\ell_p}(\mathbb{F}_n) \nrightarrow C^*_{\ell_q}(\mathbb{F}_n) \nrightarrow C^*_{\ell_2}(\mathbb{F}_n) = C^*_\lambda(\mathbb{F}_n). \]

Taking adjoint, we get completely isometric inclusions
\[ C^*_{\ell_2}(\mathbb{F}_n) = C^*_\lambda(\mathbb{F}_n)^* \hookrightarrow C^*_{\ell_q}(\mathbb{F}_n)^* \subseteq C^*_{\ell_p}(\mathbb{F}_n)^* \subseteq C^*(\mathbb{F}_n)^* = B(\mathbb{F}_n). \]

It is interesting to know whether

the operator dual $C^*_{\ell_p}(\mathbb{F}_n)^*$ of $C^*_{\ell_p}(\mathbb{F}_n)$ is finitely representable in \{\(T_n\}\}

for some (or for all) $p \in (2, \infty]$. 
Thank you for your attention!