Property (T) and Haagerup property for quantum groups – a global point of view
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Unitary representations

$G$ – locally compact group

A (unitary, strongly continuous) representation $\pi$ of $G$ on a Hilbert space $H$

- contains an invariant vector if

$$\exists \xi \in H, \|\xi\| = 1 \quad \forall g \in G \quad \pi(g)\xi = \xi;$$

- contains almost invariant vectors if

$$\exists \xi_i \in H, \|\xi_i\| = 1 \quad \forall g \in G \quad \pi(g)\xi_i - \xi_i \to 0$$

(uniformly on compact subsets)

- is mixing if

$$\forall \xi, \eta \in H \quad \langle \xi, \pi(\cdot)\eta \rangle \in C_0(G);$$

- is ergodic if it does not contain an invariant vector;

- is weakly mixing if $\pi \otimes \bar{\pi}$ is ergodic.
Haagerup property and property (T)

**Definition**

$G$ has the **Haagerup property** (HAP) if it admits a mixing representation with almost invariant vectors.

**Definition**

$G$ has the **Kazhdan Property (T)** if its every representation with almost invariant vectors contains an invariant vector.

Amenable groups have HAP; $G$ has both HAP and property (T) if and only if $G$ is compact
General notations

\( \mathbb{G} \) – a locally compact quantum group à la Kustermans-Vaes

\( L^\infty(\mathbb{G}) \) – a von Neumann algebra equipped with the coproduct

\[ \Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G}) \]

carrying all the information about \( \mathbb{G} \)

\( C_0(\mathbb{G}) \) – the corresponding (reduced) C*-object, \( C_b(\mathbb{G}) := M(C_0(\mathbb{G})) \)

\( C'_0(\mathbb{G}) \) – the universal version of \( C_0(\mathbb{G}) \)

\( L^2(\mathbb{G}) \) – the GNS Hilbert space of the right invariant Haar weight on \( \mathbb{G} \)

\( \mathcal{W}^G \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G})) \) – the multiplicative unitary associated to \( \mathbb{G} \):

\[ \Delta(f) = \mathcal{W}^G(f \otimes 1)(\mathcal{W}^G)^*, \quad f \in L^\infty(\mathbb{G}). \]

Note the inclusions

\[ C_0(\mathbb{G}) \subset C_b(\mathbb{G}) \subset L^\infty(\mathbb{G}) = C_0(\mathbb{G})'' \]
Dual quantum groups

Each LCQG $G$ admits the dual LCQG $\hat{G}$.

$L^\infty(\hat{G})$, $C_0(\hat{G})$ – subalgebras of $B(L^2(G))$

$W^G \in M(C_0(\hat{G}) \otimes C_0(G))$ and

$$W^\hat{G} = (\sigma(W^G))^*$$

In particular for $G$ – locally compact group

$$L^\infty(\hat{G}) = VN(G)$$

$$C_0(\hat{G}) = C_r^*(G), \quad C_0^u(\hat{G}) = C^*(G)$$
Further properties of LCQGs

Definition

A locally compact quantum groups $\mathbb{G}$ is

- **compact** if $C_0(\mathbb{G})$ is unital (equivalently the Haar weights are finite);
- **discrete** if $\hat{\mathbb{G}}$ is compact;
- **unimodular** if the left and right Haar weights coincide;
- **of Kac type** if the so-called scaling group is trivial (the antipode is a bounded map);
- **amenable** if $L^\infty(\mathbb{G})$ admits a bi-invariant mean;
- **coamenable** if the universal and reduced algebras $C_0(\mathbb{G})$ and $C^u_0(\mathbb{G})$ are naturally isomorphic;
- **second countable** if $C_0(\mathbb{G})$ is separable.
Some examples of locally compact quantum groups

- locally compact groups (all coamenable);
- duals of locally compact groups (all amenable);
- quantum deformations of classical Lie groups: for example $SU_q(2)$, quantum $ax + b$, $E_q(2)$ (amenable and coamenable, usually not Kac);
- quantum symmetry groups: quantum permutation groups $S_n^+$, quantum automorphism groups of Wang $G_{\text{aut}}(M_n)$, quantum orthogonal groups $O_n^+$ (mostly non-coamenable, mostly Kac).
Representations of LCQGs

Definition

A (unitary) representation of $G$ on a Hilbert space $H$ is a unitary $U \in M(C_0(G) \otimes K(H))$ such that

$$(\Delta \otimes \iota)(U) = U_{13} U_{23}.$$

The operators $(\iota \otimes \omega_{\xi,\eta})(U) \in C_b(G)$, where $\xi, \eta \in H$, are called coefficients of $U$.

Representations of $G$ are in a 1-1 correspondence with $C^*$-representations of $C_0^u(\hat{G})$.

One can also tensor representations of $G$ ($U \boxtimes V$), take direct sums ($U \oplus V$) and pass to a contragredient representation $U^c$. 
Representations of LCQGs – continued

Definition

A representation $U$ of $G$ is *mixing* if all its coefficients belong to $C_0(G)$. It *has almost invariant vectors* if there exists a net of unit vectors $(\xi_i)_{i \in I}$ such that for all $a \in C_0(G)$

$$U(a \otimes \xi_i) - a \otimes \xi_i \to 0$$

– equivalently for all $b \in C_0^u(\hat{G})$

$$\phi_U(b)\xi_i - \hat{\epsilon}(b)\xi_i \to 0 \text{ strictly.}$$

The multiplicative unitary $W^G$ plays the role of the left regular representation of $G$ on $L^2(G)$; it is mixing.
Definitions and first facts

Definition
A locally compact quantum group $\mathbb{G}$ has the Haagerup property (HAP) if it admits a mixing representation containing almost invariant vectors.

Definition
A locally compact quantum group $\mathbb{G}$ has Kazhdan Property (T) if its every representation containing almost invariant vectors contains an invariant vector.

Proposition
If $\hat{\mathbb{G}}$ is coamenable, then $\mathbb{G}$ has HAP. In particular, amenable discrete quantum groups have HAP. $\mathbb{G}$ is compact if and only if it has both HAP and Property (T).
Space of representations

\( \mathbb{G} \) – second countable locally compact quantum group, \( H \) – fixed infinite dimensional separable Hilbert space. Then \( \text{Rep}_G(H) \) is a Polish space with a natural (‘point-weak’) topology. It is equipped with two natural operations: direct sum (finite or countable) and tensoring (after we fix some unitary identifications of \( H \otimes H \) with \( H \), etc.).

**Lemma (DFSW)**

Suppose \( \mathcal{R} \subset \text{Rep}_G(H) \)

1. is stable under unitary equivalence;
2. is stable under tensoring with any \( V \in \text{Rep}_G(H) \);
3. contains a representation with almost invariant vectors.

Then \( \mathcal{R} \) is dense in \( \text{Rep}_G(H) \).
HAP and density of mixing representations

**Theorem (DFSW)**

A second countable locally compact quantum group $G$ has HAP if and only if the set of mixing representations is dense in $\text{Rep}_G(H)$.

**Proof.**

$\Leftarrow$

Approximate the trivial representation with mixing ones and take their direct sum (still mixing!).

$\Rightarrow$

Use the lemma with $\mathcal{R}$ – mixing representations.

Ideas go back to the work of Halmos for $\mathbb{Z}$. 
Theorem (Kerr-Pichot, 2012)
Let $G$ – classical locally compact group, second countable. Then $G$ does not have Property (T) if and only if weakly mixing representations are dense in $\text{Rep}_G(H)$.

We want to show the same for (a class of) quantum groups. Recall: $U \in \text{Rep}_G(H)$ is weakly mixing if $U \oplus U^c$ is ergodic.

- when is the class of weakly mixing representations stable under tensoring?
- when not (T) means that there is a weakly mixing representation with almost invariant vectors?
Weak mixing representations revisited

Classically: \( U \) is not weakly mixing if and only if \( U^\perp U^c \) contains an invariant vector if and only if \( U \) contains a finite dimensional subrepresentation.

More generally: \( U^\perp V^c \) contains a fixed vector if and only if \( U \) and \( V \) contain the same finite dimensional subrepresentation.

**Lemma (Chen+Ng 2015, see also Kyed+Sołtan, Viselter)**

If \( G \) is of Kac type, then a representation of \( G \) is weakly mixing if and only if it does not contain a finite dimensional subrepresentation; hence then the class of weakly mixing representations is stable under tensoring.
Weak mixing representations with almost invariant vectors

Let us contradict the statement: there is a representation of $G$ which is weakly mixing and has almost invariant vectors.

**Definition**

$G$ has Property $(T)^{1,1}$ (of Bekka and Valette) if for every representation $U$ of $G$ with almost invariant vectors $U \otimes U^c$ has a fixed vector.

Obviously $(T) \Rightarrow (T)^{1,1}$.

**Theorem (Bekka and Valette)**

For classical groups $(T) \iff (T)^{1,1}$.

This gives the result of Kerr and Pichot.
Discrete quantum groups with low duals

$\Gamma$ – discrete quantum group. Then

$$c_0(\Gamma) = \bigoplus_{i \in I} M_{n_i}$$

**Definition**

$\Gamma$ as above has a low dual if $\sup_{i \in I} n_i < \infty$.

In other words, we have a uniform bound on the size of irreducible representations of the compact quantum group dual to $\Gamma$. 
Main theorem

**Theorem (DSV, 2016)**

Let $\Gamma$ – discrete unimodular second countable quantum group with a low dual. Then $\Gamma$ has Property (T) if and only if it has Property $T^{1,1}$.

**Corollary (DSV, 2016)**

Let $\Gamma$ – discrete unimodular second countable quantum group with a low dual. Then $\Gamma$ does not have (T) if and only if weakly mixing representations form a dense $G_\delta$-set in $\text{Rep}_\Gamma(H)$. 
Main theorem – ingredients of the proof

**Problem**: assume $\Gamma$ does not have (T). Construct a representation $U$ of $\Gamma$ with almost invariant vectors such that $U \oplus U^c$ contains no invariant vector.

**Idea** (Jolissaint): use non (T) of $\Gamma$ to construct a semigroup of states on the algebra $C(\hat{\Gamma})$ with particular properties
Main theorem – ingredients of the proof

Actual ingredients:

- development of the notion of Kazhdan pairs for quantum groups with Property (T);
- application of this to showing that if a locally compact $\mathbb{G}$ is of Kac type and does not have (T) then one can find a net of positive-definite normalised positive elements in $C_b(\mathbb{G})$ which converge to 1 strictly, but not in norm;
- use of Yukio Arano’s work on central Property (T) to show that for discrete unimodular case one can choose the elements above in the centre;
- construction of a strongly unbounded symmetric generating functional $L$ on $\hat{\Gamma}$;
- using $L$ to generate a convolution semigroup of states $\mu_t$ on $C^u(\hat{\Gamma})$;
- building out of $\mu_t$ ‘symmetric’ GNS representations of $C^u(\hat{\Gamma})$ (thus self-contragredient representations $U_t$ of $\Gamma$);
- showing that the representations $U_t \oplus U_t$ cannot contain invariant vectors (and only here the low dual assumption comes in), and concluding by another Kazhdan pair argument.
Other consequences of \((T)\) ⇐⇒ \((T)^{1,1}\)

### Theorem (DSV, 2016)

Let \(\Gamma\) – discrete unimodular second countable quantum group with a low dual. Then the following are equivalent:

1. \(\Gamma\) has Property \((T)\);
2. for every action of \(\Gamma\) on a von Neumann algebra invariant states are limits of \textit{normal} invariant states;
3. every ergodic action of \(\Gamma\) on a von Neumann algebra preserving a faithful normal state is strongly operator ergodic (i.e. asymptotically invariant nets of elements in the von Neumann algebra are trivial).

These generalize classical results of Li, Ng, Connes and Weiss.