1. Problem 1 Show that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

using an $\epsilon - \delta$ proof.

Solution: One can see that the following inequalities are true for values close to zero, both positive and negative.

$$|\sin(x)| < |x|$$
$$|x| < |\tan(x)|$$

This in turn implies that

$$|\cos(x)| < |\frac{\sin(x)}{x}| < 1$$

On the interval $(-\pi/2, \pi/2)$, this implies

$$\cos(x) < \frac{\sin(x)}{x} < 1$$

Subtracting 1 from both sides, we have

$$\cos(x) - 1 < \frac{\sin(x)}{x} - 1 < 0$$

Taking absolute values, again, we have

$$|\frac{\sin(x)}{x} - 1| < |1 - \cos(x)|$$

This step is important, since we can show that $|1 - \cos(x)|$ goes to zero as x does, that is, the right hand side can be found in terms of δ.

Since

$$|1 - \cos(x)| = |\frac{1 - \cos^2(x)}{1 + \cos(x)}| = \frac{\sin^2(x)}{1 + \cos(x)} \leq x^2$$

Putting this together we have

$$|\frac{\sin(x)}{x} - 1| < x^2$$

Therefore, if $|x - 0| < \delta$ then

$$|f(x) - L| = |\frac{\sin(x)}{x} - 1| \leq |x|^2 \leq \delta^2$$

Summing up, if $|x| \leq \delta = \sqrt{\epsilon}$ then

$$|f(x) - L| = |\frac{\sin(x)}{x} - 1| \leq |x|^2 \leq \delta^2 = \epsilon$$

2. Problem 2 Using the results of the previous problem, show that

$$\lim_{x \to 0} \frac{\sin(\sin(x))}{x}$$

exists.

Solution: The easiest way is to write the problem as

$$\lim_{x \to 0} \frac{\sin(\sin(x))}{x} = \lim_{x \to 0} \frac{\sin(\sin(x))}{\sin(x)} \lim_{x \to 0} \frac{\sin(x)}{x}$$

Let $u = \sin(x)$, then we have

$$\lim_{x \to 0} \frac{\sin(\sin(x))}{x} = \lim_{u \to 0} \frac{\sin(u)}{u} \lim_{x \to 0} \frac{\sin(x)}{x} = (1)(1) = 1$$

3. Problem 3 Show that

$$\lim_{x \to 0} \sin(1/x)$$

does **not** exists, using an $\epsilon - \delta$ proof.

Solution: The easiest way is a **proof by contradiction**.

Suppose the limit did exist, then there would be an L such that given an $\epsilon > 0$, then $|x| < \delta$ would imply $|\sin(1/x) - L| < \epsilon$.

Choose an $\epsilon > 0$. Find the δ, depending on ϵ. We can find an x-value, e.g. $x_1 = 1/(N\pi)$ such that $|x_1| < \delta$, so that

$$|\sin(1/x_1) - L| = |\sin(N\pi) - L| < |0 - L| < \epsilon$$

Similarly, we can find an x-value, e.g. $x_2 = 1/((2N + 1/2)\pi)$ so that $|x_2| < \delta$, so that

$$|\sin(1/x_2) - L| = |\sin((2N + 1/2)\pi) - L| = |1 - L| < \epsilon$$

This is a problem, since if $\epsilon < 1/2$, L can't be close to both 0 and 1!

Intuitively, $\sin(1/x)$ oscillates to rapidly near $x = 0$. It takes on values near -1, 0, +1, arbitrarily close to $x = 0$ so it cannot approach a limit...

4. Problem 4 Show that

$$\lim_{x\to\infty}[\sqrt{x+1}-\sqrt{x}]=0$$

Note: To show that

$$\lim_{x\to\infty} f(x)=L$$

we must show that given any $\epsilon > 0$, we can find an N, depending on ϵ, such that

$$|x|>N \implies |f(x)-L|<\epsilon$$

The first step is to multiply by the conjugate

$$\lim_{x\to\infty}[\sqrt{x+1}-\sqrt{x}] = \lim_{x\to\infty}[\sqrt{x+1}-\sqrt{x}][\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}]$$

$$= \lim_{x\to\infty}\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}}$$

The critical observation is that this can be estimated in terms of N.

$$\frac{1}{\sqrt{x+1}+\sqrt{x}} < 2\frac{1}{\sqrt{x}} < 2\frac{1}{\sqrt{N}}$$

So if $|x|>N>4/\epsilon^2$, then

$$|\frac{1}{\sqrt{x+1}+\sqrt{x}}| < |\frac{2}{\sqrt{x}}| < \frac{2}{\sqrt{N}}| < \epsilon$$

This gives the relationship between N and ϵ explicitly.

5. Problem 5 The Fibonacci numbers are **defined** by the relationship

$$a_{n+1} \equiv a_n + a_{n-1}, n = 1...\infty$$

with $a_0 = 1, a_1 = 1$. What we want to show is that

$$a(n) = a_n = \frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^n + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^n$$

for **all integer values of n**. This is an indication that we must use induction.

First we show that it is true for $n = 1$.

$$a(1) = \frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^0 + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^0 = \frac{5+\sqrt{5}}{10} + \frac{5-\sqrt{5}}{10} = 1$$

which is true. Now asssume that is true for $n \leq k$, that is

$$a(k) = \frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^k + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^k$$

$$a(k-1) = \frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^{k-1} + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^{k-1}$$

Then we examine

$$a(k+1) = a(k) + a(k-1) =$$

$$\frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^k + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^k$$

$$+\frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^{k-1} + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^{k-1}$$

and by combining common terms,

$$= \frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^{k-1}(1 + \frac{1+\sqrt{5}}{2}) + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^{k-1}(1 + \frac{1-\sqrt{5}}{2})$$

The crucial observation is that

$$(\frac{1+\sqrt{5}}{2})^2 = \frac{1+2\sqrt{5}+5}{4} = 1 + (\frac{1+\sqrt{5}}{2})$$

so

$$(\frac{1+\sqrt{5}}{2})^{k+1} = (\frac{1+\sqrt{5}}{2})^{k-1} + (\frac{1+\sqrt{5}}{2})^k$$

Therefore,

$$a(k+1) = \frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^{k+1} + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^{k+1}$$

To get the asymptotics, note that

$$\frac{a_n}{\frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^n}$$

$$= 1 + \frac{5-\sqrt{5}}{5+\sqrt{5}}(\frac{1-\sqrt{5}}{1+\sqrt{5}})^n$$

and since

$$|\frac{1-\sqrt{5}}{1+\sqrt{5}}| < 1$$

$$|\frac{1-\sqrt{5}}{1+\sqrt{5}}|^n \to 0$$

hence

$$\frac{a_n}{\frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^n} \to 1$$

Therefore $c = \frac{5+\sqrt{5}}{10}$ and $r = \frac{1+\sqrt{5}}{2}$.

6. Problem 6 Show that one can compute π by means of the infinite series

$$\pi = 4 * (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...)$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

Solution: The function $\tan^{-1}(x)$ can be written as

$$\tan^{-1}(x) = \int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + ...)$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

Substituting $x = 1$, we have

$$\pi/4 = \tan^{-1} 1 = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...$$

7. Problem 7 Show that the non-alternating series

$$\sum_{n=0}^{\infty} \frac{1}{2n+1}$$

diverges. One can do this several ways. The direct approach is to group terms. If we take k successive terms, staring with $\frac{1}{2N+1}$ we have (taking k of the smallest terms as a lower bound)

$$\frac{1}{2N+1} + \frac{1}{2N+3} + \frac{1}{2N+5} + \frac{1}{2N+2k-1}$$

$$\geq \frac{k}{2N+2k-1}$$

This will be greater than a fixed constant, e.g. 1/4, if

$$\frac{k}{2N+2k-1} \geq \frac{1}{4}$$

that is, if $k \geq N - 1/2$. So, letting $k = N$ we have the inequality

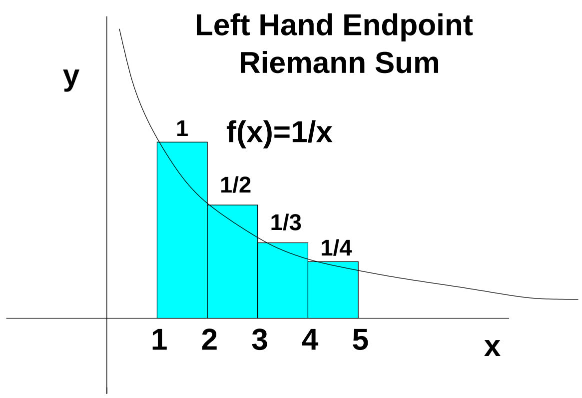
$$\frac{1}{2N+1} + \frac{1}{2N+3} + \frac{1}{2N+5} + \frac{1}{2N+2k-1} =$$

$$\frac{1}{2N+1} + \frac{1}{2N+3} + \frac{1}{2N+5} + \frac{1}{2N+2N-1}$$

$$\geq \frac{N}{4N-1} \geq \frac{1}{4}$$

Since we have an infinite number of these groups, all of which are greater than 1/4, the series diverges.

Another way of demonstrating this is to compare the series to an integral. Using the left hand rule for Riemanns sums



This implies that

$$1 + \frac{1}{3} + \frac{1}{5} + ... = \sum_{n=0}^{\infty} \frac{1}{2n+1} \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} + \frac{1}{4} + ...$$

$$= \frac{1}{2}[1 + \frac{1}{2} + \frac{1}{3} + ...] \geq \frac{1}{2}[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N}]$$

$$\geq \frac{1}{2}\int_1^N \frac{1}{x}dx = \frac{1}{2}\ln(N)$$

As $N \to \infty$, the integral (and therefore the sum) become infinite. This is known as an **integral comparison test**.

8. Problem 8 See the file probem8.pdf

9. Problem 9 Define the Heaviside function by

$$H(t) \equiv \begin{cases} 0 & \text{if t} < 0 \\ 1 & \text{if t} \geq 0 \end{cases}$$

Show, uising an $\epsilon - \delta$ argument that limit of $H(t)$ as $t \to 0$ does not exist.

Solution This is very similar to Problem 3. Again, one uses a proof by contradiction.

Suppose the limit did exist, then there would be an L such that given an $\epsilon > 0$, then $|x| < \delta$ would imply $|H(x) - L| < \epsilon$. But, for any $\delta > 0$ we can find two x values such that we must have $|H(x_1) - L| = |0 - L| < \epsilon$ and $|H(x_2) - L| = |1 - L| < \epsilon$. This leads to a contradiction if $\epsilon > 1/2$.

10. Problem 10 Show that

$$\lim_{x \to 0^+} x^x = 1$$

Solution Since $x^x = (e^{\ln x})^x = e^{x \ln x}$, we can use the properties of the limits, and the fact that e^x is continuous, to show that

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^{\lim_{x \to 0^+} x \ln x}$$

To calculate the last limit, we use L'Hôpital's rule.

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0$$

Therefore,

$$\lim_{x \to 0^+} x^x = e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1$$

You can check this out on a calculator for values very close to 0.0