Non-Euclidean Geometry

It is clear that Euclid is the "Father of Geometry." Although many of the theorems that he presented in his books The Elements were known for thousands of years, Euclid assembled them in a logical format complete with proofs.

Geometry literally means measurement of the Earth. In the time of Euclid, this was the essence of geometry. Babylonians and Egyptians were using geometry for many years to measure land in order to determine taxes. When the The Elements was written, there was a strong relationship between the physical world of the Earth and the mathematics of the day. Based on this fact, mathematicians of the time had no real reason to doubt Euclid's fifth postulate, a.k.a. the parallel postulate, as it concerned measurement of the Earth:

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Keep in mind that Euclid stated this as a postulate, not a theorem, but postulates are supposed to be "self-evident truths." Because of the wordiness of the postulate, many mathematicians believed that Euclid intended for it to be a theorem, but was never able to prove it. Additionally, he avoided using this postulate until the 29th Theorem, which further leads to doubts about his belief that this postulate was an absolute truth.

Furthermore, the Greeks of the time did have a spherical geometry which they used to measure the heavenly bodies. And, "spherics was studied for many centuries as a foundation for
the higher disciplines."¹ It is not clear why this was not seen as an opposition to Euclidean geometry. Quite probably, it is due to the fact that the foundation of geometry was tied to the earth and, therefore, was not the math that concerned the stars and planets.

For whatever reasons, many mathematicians throughout the centuries set out to prove the parallel postulate, thus making it a theorem.

The first known mathematician to undertake this cause was Ptolemy (90-168). He incorrectly believed that he had proved the postulate. His proof, which would later become known as "Playfair's Axiom" after John Playfair who published this axiom in 1795, was

Through any point in the plane, there is at most one straight line parallel to a given straight line.

Although this is, in essence, the same as the parallel postulate, Ptolemy did provide simpler wording. In most current text books, this is how the parallel postulate is stated.

The error in Ptolemy's proof was discovered by Proclus (410-485). He showed that Ptolemy's reasoning had used the fifth postulate in his proof. Upon finding the error, Proclus took up the challenge of proving it. His proof was based on the distance between two parallel lines, which is just a restatement of the fifth postulate.

Work to prove the parallel postulate came to an end for the Greeks. Whether the reason was due to the fact that Greek mathematicians had no reason to pursue it since they believed that Proclus had proven the postulate, or because Greek dominance in mathematics was coming to an end, is not known. Whatever the reason, there is no further documentation of Greek mathematicians pursuing this.

¹ (Rowe, 2006)
The next group of mathematicians to revisit the parallel postulate were the Persians, in particular, Ibn al-Haytham (Alhazen) (966-1059), Omar Khayyam (1048-1131), Nasir Eddin (1201-1274) and his son, Sadr al-Din. All of these mathematicians used a quadrilateral in their proofs.

Alhazen introduced motion into his proof, which was strictly prohibited by the Greek mathematicians. Although, he never proved the parallel postulate, he did influence other Persian mathematicians.

Omar Khayyam was one of the mathematicians influenced by the work of Alhazen. He attempted to derive the parallel postulate using his postulate, *two convergent straight lines intersect and it is impossible for two convergent straight lines to diverge in the direction in which they converge*. Using a quadrilateral, which later became known as a Saccheri quadrilateral, he tried to prove by contradiction that the summit angles of a quadrilateral with two right base angles and two congruent sides were right. He considered three cases, the angles were acute, the angles were obtuse, or the angles were right. He did prove that the angles could not be acute or obtuse, but his proof was based on false assumptions.

Nasir Eddin was the next mathematician to work on the postulate. Once again, he tried to prove the postulate by contradiction, but was unsuccessful. He had considered the ideas of elliptical and hyperbolic geometry, but further ruled them out.

Nasir's son, Sadr al-Din, considered the parallel postulate, but his biggest contribution, however, was that in 1278 he published a book which included Nasir Eddin's thoughts on the parallel postulate. In 1594, his book was published in Rome. Since Italy was the dominant
mathematical society in the 16th and 17th century, it is no wonder that the Italian mathematicians would continue the cause to prove the fifth postulate.

Of all the Italian mathematicians, the most influential to non-Euclidean geometry was (Giovanni) Gerolamo Saccheri (1667-1733). Saccheri, like Omar Khayyam, set out to prove the truth of the parallel postulate using *Reductio ad absurdum*, or proof by contradiction, using a quadrilateral. He, too, came up with three possible cases; the summit angles were obtuse, acute or right.

Before Sacherri began his proofs, he made some assumptions: *all lines are infinite in length, that in every case the exterior angle is greater than an interior angle and that a point at infinity possesses the same properties as an ordinary point*, which we now know to be false. Using these assumptions he was able to get a contradiction for the case where the summit angles are obtuse.

However, for the case where the summit angles were acute, he was never able to get a contradiction, and, unknowingly deduced several theorems which would later become theorems to hyperbolic geometry. These theorems include: through an outside point more than one parallel exists, the fourth angle of a quadrilateral with three right angles does not have to be right, the sum of the angles of a triangle is less than two right angles, and parallel lines are not everywhere equidistant. Unfortunately, Sacherri "[held the] conviction that Euclid's was the only valid geometry [and] he permitted this preconception to interfere with his logic."² His conclusion was "*the hypothesis of the acute angle is absolutely false; because it is repugnant to the nature of straight lines.*" In the year of his death, he published *Euclides ab omni naevo vindicates* (*Euclid vindicated of all blemish*).

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² (Boyer, 1991), pg. 437
Once again, the mathematical center of learning shifts, this time to the Austrian-Hungarian Empire. Although many famous mathematicians lived here, there are two whose influence on non-Euclidean geometry is well known.

The first is Johann Heinrich Lambert (1728-1777). Klügel, a professor of mathematics at the University of Helmstadt and who was familiar with Saccheri's work, "realized that Saccheri had not arrived at a contradiction but merely at results that seemed at variance with experience." He suggested to Lambert that he work on the parallel postulate. Using a quadrilateral similar to Saccheri's but with three right angles, he, too, considered three possible cases; the fourth angle is acute, obtuse or right. Unlike Saccheri, however, Lambert was unable to provide contradictions to either the case of obtuse or acute angles. He knew that the fourth angle could be obtuse if the quadrilateral were on a sphere. Using this same logic, he conjectured that given a sphere with a constant imaginary radius, which is now known as a pseudosphere, the case of the acute angle could exist. He went on to show that the area of a triangle on this surface is proportional to the defect, $\pi - (\alpha + \beta + \gamma)$. In 1766, he documented his work in "Theorie der Parallellinien", an article which was published posthumously in 1786.

At this time, more than 2000 years has passed since Euclid introduced the Elements. Non-Euclidean geometry has been considered and refuted. For the most part, it is because Euclidean geometry is so ingrained into the minds of many great thinkers. We need only to look to Rene Descartes (1596-1650), whose coordinate geometry is entrenched in Euclidean Geometry, or Immanuel Kant (1724-1804), the most influential philosopher of the 18th century whose Critique of Pure Reason contends that Euclidean geometry is a priori - true independent
of experience, or, most importantly, Isaac Newton (1643-1727), whose laws of motion used the trigonometry based on Euclidean Geometry, to see this. If Euclidean geometry is denied, is all of Newtonian physics erroneous? It is no wonder that many considered Euclidean and non-Euclidean geometry mutually exclusive. But with all the advances in the concepts of non-Euclidean geometry, the time is ripe for a revolution. The 19th century provides this revolution, beginning with, arguably, the most influential mathematician of non-Euclidean geometry, Carl Freidrich Gauss (1777-1855).

From the age of 12, Gauss believed the fifth postulate to be wrong, and was critical of Euclid for basing a whole branch of mathematics on a false assumption. At the time, he was under the tutelage of a man named John Martin Bartels.

It is unclear whether previous mathematicians outright denied the truth of the parallel postulate. If they did, they never documented it. Gauss was not under this belief. Because of this, Gauss was able to make numerous contributions, tangible and intangible, to non-Euclidean geometry, which, by the way, is a term he coined.

In 1818, Gauss was asked to complete a geodesic survey of Hanover. It is during this time he began to develop his theory of surfaces and the idea of curvature. His theory dealt only with surfaces with "continuous curvature", that is, surfaces which have a tangent plane at all points. Using his definition of curvature, he showed that the curvature of a sphere is \( k = \frac{1}{r^2} \), and that of a plane is \( k = 0 \). In 1827, he published a paper where he showed curvature was intrinsic to the surface. He also "established a relationship between curvature and the sum of the angles
of a triangle on the surface."\(^3\) In particular, he calculated that the total curvature over a geodesic triangle is \(A + B + C - \pi\). He also determined that if there is a one-to-one function between two surfaces’ curvature then length is preserved and the curvature at corresponding points is equal.

Gauss’ other tangible contribution was that of parameterization of the equations to describe a surface. This allowed him to express formulas in terms of quantities \(E, F,\) and \(G\) and he was able to show that the properties of a surface were dependent solely on these three quantities. Furthermore, he determined that if there is a one-to-one correspondence between two surfaces, then they are isometric and the geometry is the same. This was the beginning of the general theory of \(n\)-dimensional manifolds, which would be further developed by Riemann.

Gauss made great strides in non-Euclidean geometry and documented everything. Unfortunately, he refused to publish any of his ideas because he did not want to get into a controversy with other mathematicians of the time. His work on non-Euclidean geometry was discovered after his death.

Although he never published his work, Gauss’ influence on the advancement of non-Euclidean geometry was widespread. He corresponded with many prominent mathematicians discussing the parallel postulate. Through this correspondence, his ideas were disseminated to various locations around the world. Three people of particular importance are Georg Friedrich Bernhard Riemann (1826–1866), from Germany, Janos Bolyai (1802-1860), from Hungary, and Nikolai Ivanovich Lobachevsky (1792-1856), from Russia.

Riemann, who was a student of Gauss, "fueled the curved space revolution."\(^4\) On June 10, 1854, Riemann was to give a trial lecture. He

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\(^3\) (Katz, 1998), pg. 767  
\(^4\) (Mlodinow, 2001), pg. 135
gave three topics to Gauss to review. It was common at the time for the mentor to pick the first or second topic. Of course, Gauss, known to be extremely tough on students, chose Riemann's third topic "On the Hypotheses Which Lie at the Foundations of Geometry." Riemann's lecture focused on the geometry of infinitesimally small regions of space, now known as differential geometry. Although his lecture was a success, it was not published until after his death in 1868. However, his "memoir of 1854 attracted general attention to the subject of non-Euclidean geometry."\(^5\)

Directly influenced by Gauss, Janos Bolyai and Nicolai Lobachevsky, are generally considered to be the founders of non-Euclidean geometry, although that title was given to them retrospectively. During their lifetime, their work was published in obscure journals, or, as in the case of Bolyai, as an appendix to a book his father published. However, each of them, independently, logically deduced a system of geometry which was based on the fact that infinitely many lines could be parallel to another line through a point, now known as hyperbolic geometry.

Janos Bolyai was the son of Farkas Bolyai, a mathematician himself who corresponded frequently with Gauss about the "problem of the parallels." Farkas tried to discourage Janos from working on this problem, but Janos could not be dissuaded. Janos' work led him to the proof of hyperbolic geometry. In 1829, in what he dubbed the "Absolute Science of Space," Janos proved that through a point not on a line there are infinitely many lines parallel to the line. He also showed that in non-Euclidean geometry one could square a circle. His excitement in his discoveries was enormous. In a letter to his father he wrote, "I have made such

\(^5\) (Ball, 1919), pg. 485
wonderful discoveries that I am myself lost in astonishment" and "I've created something out of nothing." Unfortunately, when seeking Gauss' approval, Gauss was unable to praise Janos for doing so would be as if he was praising himself. Janos was devastated by Gauss' lack of enthusiasm, and, as a result, abandoned all work on non-Euclidean geometry.

Nicolai Lobachevsky studied in Russian and was taught by John Martin Bartels, the same man who taught Gauss. Undoubtedly, Bartels must have directed Lobachevsky's work on non-Euclidean geometry because his and Bolyai's work is so similar that even Bolyai thought that Lobachevsky had copied it. It is too great a coincidence that the two mathematicians could otherwise discover the same ideas, thousands of miles apart, with no communication.

Lobachevsky has been called the "Copernicus of Geometry," because, although he and Bolyai came to the same conclusions, Lobachevsky was the first to be published. His work was presented in an 1829 article entitled "On the Principles of Geometry," and is considered the official birth of non-Euclidean geometry.

As well as presenting many theorems, Lobachevsky developed an extensive trigonometry of the non-Euclidean plane. He also showed that as the triangle became small, the trigonometry followed that of Euclidean geometry.

Unfortunately, Bolyai's and Lobachevsky's work did not produce a geometric revolution. They published in obscure journals and mathematicians at the time had a hard time envisioning a non-Euclidean plane. They also could not see how this new geometry related to the real world. It was Hermann von Helmholtz (1821-1894) who promulgated their work into the limelight.
Helmholz was influenced by Riemann's posthumous paper. It was he who brought Reimann's work to the public by interpreting it and extending upon it. He was also able to show that Lobachevsky's geometry was a subset of Reimann's.

Although there were still skeptics of non-Euclidean geometry, the mathematical community was beginning to embrace the idea. Of these were an Italian mathematician named Eugenio Beltrami (1835–1900) and a German mathematician named C. Felix Klein (1849–1925). "The works of Beltrami and Klein that appeared between 1866 and 1871 put hyperbolic geometry on a par with Euclidean geometry."\(^6\)

Beltrami discovered that if a Tractrix curve is rotated about its axis, a surface is created that has constant negative curvature, a pseudo-sphere. The world now had a model in which to visualize non-Euclidean geometry. Additionally, he showed that the pseudo-sphere is locally isometric to the hyperbolic disc; this discovery "[changed] what was thought to be a figment of the imagination into a mathematical fact."\(^7\)

Klein's work provided a connection between projective geometry, which was the predominant geometry of research at the time, and non-Euclidean geometry. He also developed a model and, of course, the famous Klein Bottle. His most valuable contribution, however, is the fact that he was a superb teacher who gave inspiring lectures. He was a "professor of mathematics at Gottingen, and under his leadership the university became a Mecca to which students from many lands, including America, flocked."\(^8\)

Towards the end of the 19th century, the work in the various forms of geometry, Euclidean, Hyperbolic, Spherical and Elliptical, grew exponentially. Reimann had shown that there does not have to be physical evidence to support geometry. He was able to generalize all

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\(^6\) (Shenitzer, 1994)
\(^7\) (Shenitzer, 1994)
\(^8\) (Boyer, 1991), pg. 550
the forms of geometry to $n$-dimensions. He also created differential geometry, which studies the geometry of a very small region on a surface, no matter how many dimensions. It was "Riemann's work on differential geometry [that] became the cornerstone of Einstein's general theory of relativity."\(^9\)

The work in non-Euclidean geometry obviously had an effect on science, physics in particular, but it also was seen in the art world. M. C. Escher, for example, incorporated hyperbolic geometry into many of his pieces. An example is his "Angels and Demons", which uses tessellation on a hyperbolic disc model.

Over two thousand years went into the making of non-Euclidean geometry. Although it is not clear who the "father" is, whether it's Gauss, Reimann, Bolyai or Lobachevsky, what is clear is that its influence has been as revolutionary as "The Elements" were back in 300 B.C.E, albeit for different reasons.

Humans have come to realize that their view of the universe was myopic. The geometry of the Earth does not necessarily apply to the far reaches of space or to the physics of the atom. The development of the concept of $N$-dimensional geometries has propelled the development of various mathematics and theoretical physics, of which the Theory of Relativity is based. It is the theoretical math that is now being used to predicting the physical aspects. Through it all, however, "geometry has been the window to understanding our universe."\(^{10}\)

\(^9\) (Mlodinow, 2001), pg. 142
\(^{10}\) (Mlodinow, 2001) pg. 149
Bibliography


